Project 4: The "Butterfly Effect" and Intermittency

In this project we will examine the sensitivity of nonlinear dynamic systems to small changes in parameters and initial conditions. Our objective will be to show that in some cases even the most miniscule variations can produce significant effects, while in others there will be no noticeable difference at all. Edward Lorenz (who discovered the phenomenon of chaos) described this hypersensitivity to small changes as the "butterfly effect", referring to the possibility that something as insignificant as the movement of a butterfly's wings could ultimately affect global weather patterns.

We will also consider an unusual property known as *intermittency*, where the system exhibits long periods of regular behavior which are interrupted by sudden and unpredictable aperiodic bursts. The existence of such phenomena suggests that the behavior of physical systems can occasionally deviate from empirically established rules and patterns for no apparent "reason". This brings into question the classical assumption that every observable effect must have an identifiable cause.

Problem 1. Consider the second order system

$$\begin{aligned}\dot{x}_1 &= -3x_1 + 4x_1^2 - 0.5x_1x_2 - x_1^3 \\ \dot{x}_2 &= -2.1x_2 + x_1x_2\end{aligned}$$

(a) Solve the equation numerically for initial condition: $x_0 = [4; 1]$, using $t = 0 : 0.01 : 50$. Repeat this process with initial condition

$$y_0 = x_0 + 1e - 4 * [1; 1]$$

which differs from the previous one at the fourth decimal point. To evaluate the difference between these two solutions (whose components will be denoted $[x_1; x_2]$ and $[y_1; y_2]$, respectively) plot $x_1 - y_1$ as a function of time. Would you say that a small perturbation in the initial conditions produces a small change in the solution, or is the change substantial?

(b) Repeat part (a) of the problem using initial condition

$$y_0 = x_0 + 1e - 8 * [1; 1]$$

and plot $x_1 - y_1$ as a function of time. Do your results confirm your previous conclusions? Explain.

Problem 2. Consider the second order system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= (1 - x_1^2)x_2 - x_1\end{aligned}$$

(a) Solve the equation numerically for initial condition: $x_0 = [-1.5; 3.5]$, using $t = 0 : 0.01 : 10$. Repeat this process with initial condition

$$y_0 = x_0 + 1e - 4 * [1; 1]$$

and plot $x_1 - y_1$ as a function of time. Would you say that a small perturbation in the initial conditions produces a small change in the solution?

(b) Repeat part (a) of the problem using initial condition

$$y_0 = x_0 + 1e - 8 * [1; 1]$$

and plot $x_1 - y_1$ as a function of time. Do your results confirm your previous conclusions? Explain.

Problem 3. Consider the second order system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= (1 - x_1^2)x_2 - x_1 + 0.5\cos 2t \end{aligned}$$

(a) Solve the equation numerically for initial condition: $x_0 = [-1.5; 3.5]$, using $t = 0 : 0.01 : 50$. Repeat this process with initial condition

$$y_0 = x_0 + 1e - 4 * [1; 1]$$

and plot $x_1 - y_1$ as a function of time. Would you say that a small perturbation in the initial conditions produces a small change in the solution?

(b) Repeat part (a) of the problem using initial condition

$$y_0 = x_0 + 1e - 8 * [1; 1]$$

and plot $x_1 - y_1$ as a function of time (use $t = 0 : 0.01 : 10$ in this case). Do your results confirm your previous conclusions? Explain.

Problem 4. Consider the third order system

$$\begin{aligned} \dot{x}_1 &= -10x_1 + 10x_2 \\ \dot{x}_2 &= -x_1x_3 + 28x_1 - x_2 \\ \dot{x}_3 &= x_1x_2 - 2.5x_3 \end{aligned}$$

(a) Solve the equation numerically for initial condition: $x_0 = [1; 1; 1]$, using $t = 0 : 0.01 : 100$. Repeat this process with initial condition

$$y_0 = x_0 + 1e - 4 * [1; 1; 1]$$

and plot $x_1 - y_1$ as a function of time. Would you say that a small perturbation in the initial conditions produces a small change in the solution?

(b) Repeat part (a) of the problem using initial conditions

$$z_0 = x_0 + 1e - 8 * [1; 1; 1]$$

and

$$w_0 = x_0 + 1e - 12 * [1; 1; 1]$$

(we will denote the corresponding solutions by $[z_1; z_2; z_3]$ and $[w_1; w_2; w_3]$, respectively). Plot $x_1 - z_1$ and $x_1 - w_1$ as functions of time, and use these graphs to evaluate whether your previous conclusions about the system's sensitivity to initial conditions are still valid.

(c) Based on the results obtained in parts (a) and (b), would you say that this system is hypersensitive to differences in initial conditions? If not, is it in any way different from the ones considered in Problems 1-3? Explain.

Problem 5. In this problem we will consider the first order discrete system

$$x(k+1) = px(k)[1 - x(k)]$$

in which parameter p is set to equal 3.9 (recall that we already analyzed this system in Project 3).

(a) Solve the equation numerically for initial condition $x(0) = 0.5$, using $k_0 = 0$ and kvect $=$ $0 : 1 : 250$. Repeat this process with initial condition

$$y_0 = 0.5 + 1e - 4$$

and plot the difference $x - y$ using function

$$\text{plot(kvect, } x - y, \text{kvect, } x - y, \text{`.'})$$

Would you say that a small perturbation in the initial conditions produces a small change in the solution?

(b) Repeat part (a) of the problem using initial condition

$$z_0 = 0.5 + 1e - 8$$

(following our previous notation, we will denote the corresponding solution by z). Plot the difference $x - z$ as a function of k. Are your results consistent with your previous conclusions?

(c) Based on the graphs obtained in parts (a) and (b), would you say that hypersensitivity to initial conditions can occur even in a simple first order discrete system? Does this surprise you? Explain.

Problem 6. In this problem we will consider the same system as in Problem 5, and focus on values of parameter p in the range $3.82 \le p \le 3.83$.

(a) Set $p = 3.82$, and solve the equation numerically for initial condition is $x(0) = 0.5$ (with $k_0 = 0$ and kvect $= 0 : 1 : 500$). Plot $x(k)$ as a function of k, and use this graph to determine whether or not the solution displays any form of regularity.

Note: To keep the graph as simple as possible, use the command

$$\text{plot(kvect, } x)$$

instead of

$$\text{plot(kvect, } x, \text{kvect, } x, \text{'.')}$$

(b) Repeat part (a) with $p = 3.83$, and plot $x(k)$ as a function of k. How does this solution compare to the one obtained using $p = 3.82$? Explain.

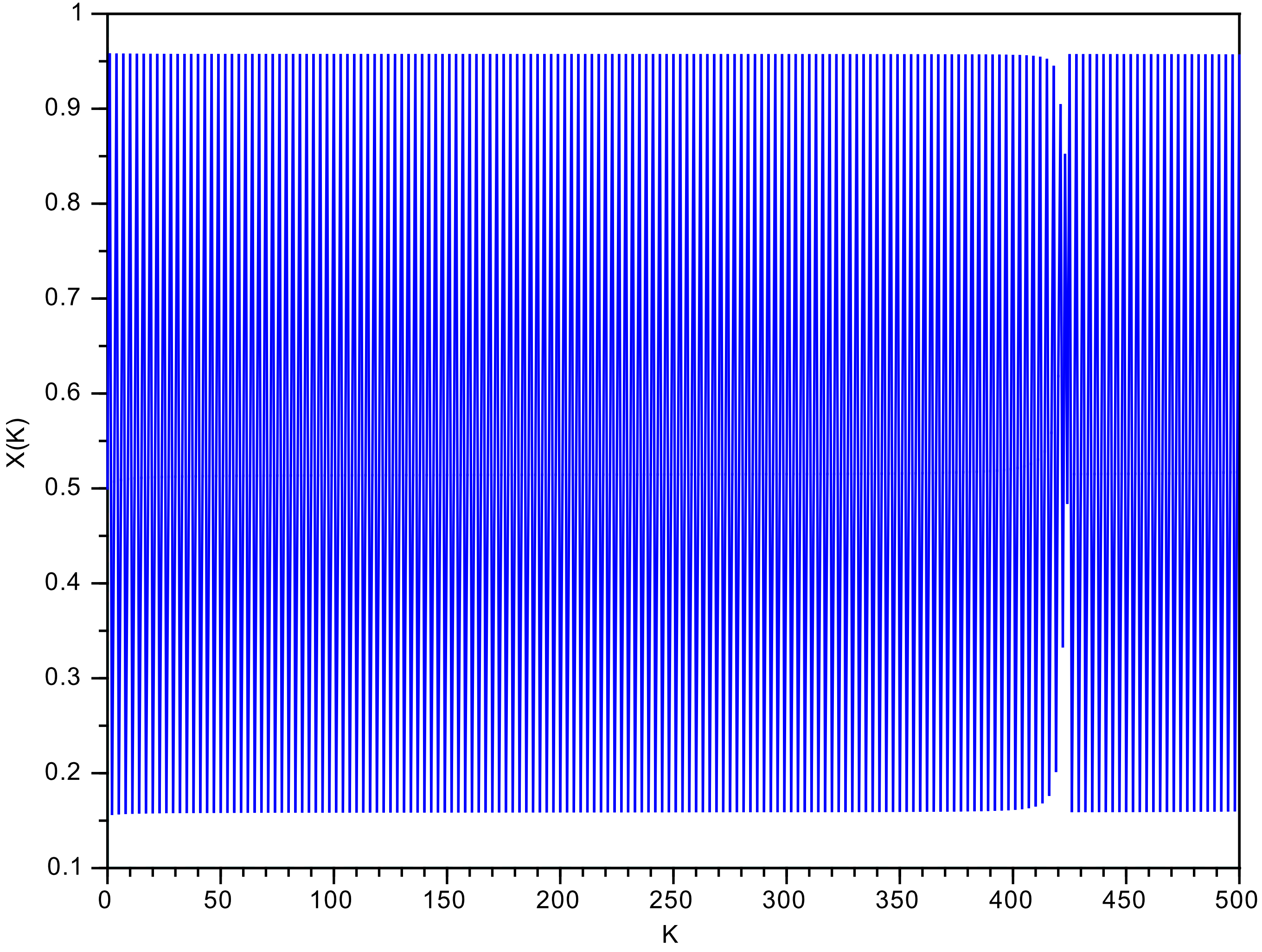
Note: For a better resolution, set the data bounds for the x-axis to [0 100].

(c) For some value p_0 between 3.82 and 3.83, the system makes a transition from random to orderly dynamic behavior. Find a value of p that is sufficiently close to p_0 to produce a graph of the form shown in Fig. 1. Adjusting the data bounds for the x-axis to [350 500] provides a closer look at the irregular segment of this solution, which is shown in Fig. 2.

Note: You will need to specify p using at least 5 decimals replicate this plot.

(d) If you were to empirically observe the behavior shown in Fig. 1 over a long period of time (where k could be interpreted as the number of months, for example), how would you interpret it? Would you say that your data conforms to a "law"? If so, how would you explain the irregularities that occur between $k = 410$ and $k = 430$? If not, how would you explain the long intervals orderly behavior?

Figure 1



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
X(K)
0
50
100
150
200
250
300
350
400
450
500
K

Figure 2