Project 2: The Four Types of Attractors

The purpose of this project is to introduce the notion of an *attractor*, which is a fundamental concept in the analysis of dynamic systems. You can think of an attractor as a sort of "magnet", which "gathers" solutions that start from different initial conditions, and draws them toward itself . In the following problems, you will have a chance to examine the four types of attractors that can arise in nonlinear dynamic systems. The most interesting kind are so-called *strange attractors*, which are a "trademark" of chaos. These geometric objects are *fractals*, which means that their dimension is *not* a whole number.

Note: Before you begin working on this project, make sure you go over Demo 2.

Problem 1. Consider the second order *linear* differential equation

$$\dot{x}_1 = x_1 - x_2$$
$$\dot{x}_2 = 3x_1 - 2.5x_2$$

(a) Solve the equation numerically for the following four initial conditions: $x_0 = [1; -1]$, $x_0 = [-1; 1]$, $x_0 = [-1; -3]$, and $x_0 = [0.5; 2]$ (use $t = 0 : 0.01 : 20$ in all cases). Plot the four solutions for $x_1(t)$ on a *single* diagram, and use this diagram to determine what happens to $x_1(t)$ when $t \to \infty$. Repeat this exercise for $x_2(t)$ (and create a separate plot for this purpose).

(b) Use the solutions obtained in part (a) to construct a *phase plot* for this system. Based on this plot, how would you describe the geometrical object to which the solutions are attracted?

Problem 2. In this problem, we will look at a different linear system, which is described as

$$\dot{x}_1 = -0.5x_1 + 4x_2$$
$$\dot{x}_2 = -4x_1 - 0.5x_2$$

(a) Solve the equation numerically for initial conditions: $x_0 = [1; 1]$ and $x_0 = [-1; -1]$ (use $t = 0 : 0.01 : 12$ in both cases). Plot the two solutions for $x_1(t)$ on a *single* diagram, and use this diagram to determine what happens to $x_1(t)$ when $t \to \infty$. Repeat this exercise for $x_2(t)$ (and create a separate plot for this purpose).

(b) Use the solutions obtained in part (a) to construct a *phase plot* for this system. Based on this plot, how would you describe the geometrical object to which the solutions are attracted?

(c) How is the phase plot obtained in this problem different from the one obtained in Problem 1? What do you think causes this difference? Can you also identify some similarities between the two plots? Explain.

Problem 3. Consider the second order *nonlinear* system

$$\dot{x}_1 = -3x_1 + 4x_1^2 - 0.5x_1x_2 - x_1^3$$
$$\dot{x}_2 = -2.1x_2 + x_1x_2$$

(a) Solve the equation numerically for the following four initial conditions: $x_0 = [4; 2]$, $x_0 = [-1; 3]$, $x_0 = [0.5; -2]$, and $x_0 = [-2; -2]$ (use $t = 0 : 0.01 : 50$ in all cases). Plot the four solutions for $x_1(t)$ on a *single* diagram, and use this diagram to determine what happens to $x_1(t)$ when $t \to \infty$. Repeat this exercise for $x_2(t)$ (and create a separate plot for this purpose). Note: For better resolution, set the data bounds for the x-axis to $[0 \;\; 10]$.

(b) Repeat part (a) for the following four initial conditions: $x_0 = [4; 1]$, $x_0 = [5; 0.5]$, $x_0 = [3; 2]$, and $x_0 = [1.5; 1.5]$ (use $t = 0 : 0.01 : 50$ once again). For better resolution, set the data bounds for the x-axis to $[0 \;\; 25]$ in both cases.

(c) Use the solutions obtained in parts (a) and (b) to construct a *phase plot* for this system (I suggest you use the following data bounds: $[-3 \;\; 6]$ for the x-axis and $[-3 \;\; 4]$ for the y-axis). Based on this plot, how would you describe the geometrical object to which the solutions are attracted? What is its dimension?

(d) In your opinion, what would be the most important difference between the phase plot obtained in part (c), and the ones you obtained in Problems 1 and 2? Explain.

Problem 4. In this problem, we will examine a *time varying* nonlinear system of the form

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = (1 - x_1^2)x_2 - x_1 + a\cos bt$$

where a and b are parameters.

(a) Set $a = 0$ and $b = 2$, and solve the equation numerically for the following three initial conditions: $x_0 = [1; 1]$, $x_0 = [-1.5; 3.5]$ and $x_0 = [-2.5; -2.5]$ (using $t = 0 : 0.01 : 100$). Plot the three solutions for $x_1(t)$ on a *single* diagram, and use this diagram to determine what happens to $x_1(t)$ when $t \to \infty$. Repeat this exercise for $x_2(t)$ (and create a separate plot for this purpose). Note: For better resolution, set the data bounds for the x-axis to $[0 \;\; 25]$.

(b) Use the solutions obtained in part (a) to construct a *phase plot* for this system. Based on this plot, how would you describe the geometrical object to which the solutions are attracted? What is its dimension?

(c) The attractor obtained in part (b) is clearly different from the ones obtained in Problems 1-3. How would you correlate this difference with the behavior of solutions $x_1(t)$ and $x_2(t)$ when $t \to \infty$? Explain.

Problem 5. Let us once again consider the system given in Problem 4, this time with $a = 0.5$ and $b = 2$.

(a) Solve the equation numerically for the same three initial conditions as in Problem 4, using $t = 0 : 0.05 : 1,000$. Plot the three solutions for $x_1(t)$ on a *single* diagram, and use this diagram to determine what happens to $x_1(t)$ when $t \to \infty$. Repeat this exercise for $x_2(t)$ (and create a separate plot for this purpose). How do your solutions differ from the ones obtained in Problem 4? Note: For better resolution, set the data bounds for the x-axis to $[0 \;\; 25]$.

(b) Use the solutions obtained in part (a) to construct a *phase plot* for this system. Based on this plot, how would you describe the geometrical object to which the solutions are attracted? What is its dimension?

(c) Compare this attractor to the one obtained in Problem 4. How would you explain the differences between them? In answering this question, consider how the solutions of these two systems behave when $t \to \infty$.

Problem 6. In this problem, we will examine a third order nonlinear system of the form

$$\begin{aligned}\dot{x}_1 &= -x_2 - x_3 \\ \dot{x}_2 &= x_1 + 0.2x_2 \\ \dot{x}_3 &= x_1 x_3 - 5.7x_3 + 0.2\end{aligned}$$

(a) Solve the equations numerically for the following three initial conditions: $x_0 = [0;0;0]$, $x_0 = [-12;-10;-1]$ and $x_0 = [-2;-6;0]$ (using $t = 0 : 0.01 : 1,000$). Plot the three solutions for $x_1(t)$ on a *single* diagram, and use this diagram to determine what happens to $x_1(t)$ when $t \to \infty$. Repeat this exercise for $x_2(t)$ and $x_3(t)$ (create a separate plot for each variable). Do you see any sort of regularity emerging over time, or do these solutions look random?

Note: You should make your conclusions based on the *entire* simulation interval $[0 \;\; 1,000]$. However, in the plots that you submit, set the data bounds for the x-axis to $[0 \;\; 50]$ - they will look nicer.

(b) In order to establish whether or not the solutions obtained in part (a) exhibit any sort of regularity (such as long-term periodicity, for example) it is helpful to look at the differences between these functions. To keep things simple, in the following, we will focus on the solutions that correspond to $x_0 = [0;0;0]$ and $x_0 = [-12;-10;-1]$ (we will denote their components by $[x_1;x_2;x_3]$ and $[y_1;y_2;y_3]$, respectively). Plot the differences $x_1 - y_1$, $x_2 - y_2$, and $x_3 - y_3$ as functions of time on three separate graphs, and use this information to determine whether the two solutions have any common features when $t \to \infty$. If they do, this would suggest the existence of some kind of "hidden" pattern.

(c) To further investigate the dynamic behavior of this system, construct a *phase plot* using the solutions obtained in part (a). Since you are working with a three dimensional system, it will be helpful to use the command

param3d(z1,z2,z3)

where $[z_1;z_2;z_3]$ corresponds to the solution with initial condition $x_0 = [-2;-6;0]$. This command produces a 3-dimensional phase plot, which you can view from different angles. To adjust the viewing angle, click on Axes Properties, and select the Viewpoint option. For the best view, replace angle 35 with 135.

(d) For a somewhat different view of the phase plot, it is useful to examine its projection onto the $x_1 - x_2$ plane. You can do this using the following command:

plot(x1,x2,0,0,'.',y1,y2,-12,-10,'.',z1,z2,-2,-6,'.')

Based on this plot and the one obtained in part (c), would you say that the system has an attractor? If so, how would you describe its geometrical properties?