Project 3: Order, Randomness and What Lies In Between

The purpose of this project is to demonstrate that order and randomness needn't be mutually exclusive, and that they can actually *coexist* in certain types of dynamic systems. We will initially examine this property in the context of chaos theory, and consider how seemingly random dynamic behavior can arise in models that are completely deterministic. We will then look at a somewhat different kind of system, in which randomness is introduced from the "outside". Our objective in this case will be to determine whether or not the solutions of such a system exhibit some sort of order.

Note: Before you begin working on this project, make sure you go over Demos 3 and 4.

Problem 1. Consider the third order nonlinear system

$$\begin{aligned}\dot{x}_1 &= -10x_1 + 10x_2\\ \dot{x}_2 &= -x_1x_3 + 28x_1 - x_2\\ \dot{x}_3 &= x_1x_2 - ax_3\end{aligned}$$

where a is a parameter. Note that there is nothing uncertain about this system once the value of a is specified, so we have every reason to expect orderly and predictable dynamic behavior.

(a) Set $a = 5$, and solve the equations numerically for the following two initial conditions: $x_0 = [1; 1; 1]$ and $x_0 = [-1; -1; -1]$ (use $t = 0 : 0.01 : 100$ in both cases). Plot the two solutions for $x_1(t)$ on a single diagram, and use this diagram to determine what happens to $x_1(t)$ when $t \to \infty$. Repeat this for $x_2(t)$ and $x_3(t)$, creating a separate plot for each variable. Do you see any sort of regularity emerging over time, or do these solutions look random? Explain.

Note: For better resolution, set the data bounds for the x-axis to $[0\ \ 20]$.

(b) Based on the solutions obtained in part (a), construct a three dimensional *phase plot* using the command

$$\text{param3d}(x_1, x_2, x_3)$$

(in this command x_1, x_2 and x_3 represent components of the solution that corresponds to initial condition $x_0 = [1; 1; 1]$).

Note: A good view of the system trajectory can be obtained by setting the rotation angles in the Viewpoint option to -150 and 60.

(c) Obtain the projection of the phase plot onto the $x_1 - x_3$ plane using the command:

$$\text{plot}(x_1, x_3, 1, 1, \text{`.'}, y_1, y_3, -1, -1, \text{`.'})$$

where y_1 and y_3 represent the first and third components of the solution with initial condition $x_0 = [-1; -1; -1]$. Based on this plot and the one obtained in part (b), would you say that the system exhibits orderly behavior? Explain.

Problem 2. Consider the same system as in Problem 1, this time with parameter a set to $a = 2.5$.

(a) Solve the equations numerically for $x_0 = [1; 1; 1]$ and $x_0 = [-1; -1; -1]$ using $t = 0 : 0.01 : 100$. Plot the two solutions for $x_1(t)$ on a single diagram, and use this diagram to determine what happens to $x_1(t)$ when $t \to \infty$. Repeat this for $x_2(t)$ and $x_3(t)$ (create a separate plot for each variable). Do you see any sort of regularity emerging over time, or do these solutions look random?

Note: For better resolution, set the data bounds for the x-axis to $[0 \;\; 35]$.

(b) Examine the *difference* between the two solutions using commands plot$(t, x_1 - y_1)$, plot$(t, x_2 - y_2)$ and plot$(t, x_3 - y_3)$, where $[x_1; x_2; x_3]$ denotes the components of the solution with initial condition $x_0 = [1; 1; 1]$, and $[y_1; y_2; y_3]$ corresponds to $x_0 = [-1; -1; -1]$. Do these plots support the conclusions you reached in part (a)? Explain.

(c) Use the solutions obtained in part (a) to construct a three dimensional *phase plot* for this system, and its projection onto the $x_1 - x_3$ plane. Based on these two plots, would you characterize the solutions as orderly or random? Explain your reasoning.

Note: As in Problem 1b, a good view of the system trajectory can be obtained by setting the rotation angles in the Viewpoint option to -150 and 60.

(d) Compare the results obtained in Problem 1 (with $a = 5$) and Problem 2 (with $a = 2.5$), and describe how changing parameter a affects the system dynamics. Is it possible for the same system to exhibit *both* regular *and* random behavior, depending on the choice of a single parameter? If so, do you think regularities are easier to detect by looking at the solution as a function of time, or by looking at the phase plots? Explain.

Problem 3. Consider the first order discrete system

$$x(k+1) = px(k)[1 - x(k)]$$

where p is a known parameter, and the initial condition is $x(0) = 0.5$. As in Problems 1 and 2, this system is completely deterministic once the parameter value is specified.

(a) Solve the equation numerically for $p = 2.9$, $p = 3.4$ and $p = 3.9$ (using $k_0 = 0$ and kvect $= 0 : 1 : 100$ in all cases). Plot each solution separately, and determine what happens to $x(k)$ when $k \to \infty$. Does the solution show any regularities, or does it appear random?

Note: For better resolution, set the data bounds for the x-axis to $[0 \;\; 50]$ for $p = 2.9$ and $p = 3.4$, and use $[0 \;\; 100]$ for $p = 3.9$.

(b) Compare the three sequences obtained in part (a), and describe how changing parameter p affects the system dynamics. Can this simple system exhibit both regular and random behavior, depending on the choice of a single parameter? Explain.

(c) Construct a *random* sequence of points using command

$$z = \text{rand}(1, 101)$$

and plot $z(k)$ as a function of k. Do you see any difference between this plot and the one obtained in part (a) for $p = 3.9$? Explain.

(d) To gain a somewhat different perspective on the solutions obtained in part (a), plot $x(k+1)$ versus $x(k)$ for the case when $p = 3.9$. Do you notice any emergent patterns in this graph, or is it just a set of randomly scattered points?

(e) Solve the system using kvect $= 0 : 1 : 10,000$, and plot $x(k+1)$ versus $x(k)$ once again. Is your graph consistent with the one you obtained in part (d)? Would you characterize it as orderly or random?

(f) For the sake of comparison, plot $z(k+1)$ versus $z(k)$ for the random sequence obtained in part (c). Does this graph resemble the one you obtained in part (d), or do you notice any differences?

(g) Generate a pair of random sequences using commands

$$z = \text{rand}(1, 10001)$$

and

$$z = \text{rand}(1, 100001)$$

Plot $z(k+1)$ versus $z(k)$ in both cases (on separate graphs), and determine whether an orderly structure emerges.

(h) Based on the results obtained in parts (e), (f) and (g), how would you characterize the differences between the solution for $p = 3.9$ and a purely random sequence? Are there any similarities? Explain.

Problem 4. In this problem, we will examine the second order discrete system

$$x_1(k+1) = 0.5\,[x_1(k) + c(1)]$$
$$x_2(k+1) = 0.5\,[x_2(k) + c(2)]$$

where $c(1)$ and $c(2)$ are *randomly* chosen in each step. Unlike the systems that we examined in Problems 1 - 3, this one is clearly *not* deterministic, so it will therefore be interesting to see whether its dynamic behavior shows any elements of order.

(a) Solve the equations numerically for $x_1(0) = x_2(0) = 0.5$, using $k_0 = 0$ and kvect $= 0 : 1 : 100$. In each iteration, set the pair $c = [c(1); c(2)]$ to equal *one* of the following three combinations: $c = [0; 0]$, $c = [0.5; 1]$ or $c = [1; 0]$ (the choice should be made randomly, as in Demo 4). Using the obtained solution, plot $x_2(k)$ versus $x_1(k)$. Does this graph look like a collection of randomly scattered points, or do you see an orderly pattern developing? Explain.

(b) Repeat part (a) of the problem using kvect $= 0 : 1 : 1,000$, kvect $= 0 : 1 : 10,000$ and kvect $= 0 : 1 : 100,000$ (produce a separate plot for each case). Based on these results, do you think a system with random elements can nevertheless exhibit orderly dynamic behavior? If so, how is that order manifested?

(c) Since the properties of this system depend on how $c(1)$ and $c(2)$ are chosen, it is of interest to examine what will change if we pick these two values in a slightly different way. In order to do that, set the pair $c = [c(1); c(2)]$ to equal one of the following three combinations: $c = [1; 1]$, $c = [1; 2]$ or $c = [2; 2]$ (the choice should once again be made randomly). Pick $x_1(0) = x_2(0) = 1.5$ as the initial conditions, and solve the equations numerically for using $k_0 = 0$ and kvect $=$

$0 : 1 : 100$. Plot $x_2(k)$ versus $x_1(k)$ and describe how this graph compares to the one you obtained in part (a).

(d) In order to increase the number of points in your solution, repeat the simulation for kvect $= 0 : 1 : 1,000$, kvect $= 0 : 1 : 10,000$ and kvect $= 0 : 1 : 100,000$ (produce a separate plot for each case). How do these graphs compare to what you obtained in part (b) of the problem?

(e) Based on the results that you obtained in parts (a)-(d), can you draw any general conclusions about the behavior of this system? Does a combination of order (in the form of equations) and disorder (in the form of random inputs $c(1)$ and $c(2)$) necessarily produce random solutions, or is there an emergent pattern? Explain.