Demo 2: Phase Plots and Attractors

In this demo we will consider a somewhat different way of describing attractors, which does not involve plotting the solution as a function of time. In order to do that, we will focus on the following second order system

$$\dot{x}_1 = -2x_1 + 0.5x_2^2$$
$$\dot{x}_2 = x_1x_2 - 2x_2 + p$$

where p represents a parameter. The equilbria of this system can be found by solving the system

$$0 = -2x_1 + 0.5x_2^2$$
$$0 = x_1x_2 - 2x_2 + p$$

In the special case when $p = 0$ this is pretty easy to do, since the second equation can be expressed as

$$0 = x_1x_2 - 2x_2 = x_2(x_1 - 2)$$

This allows for only two possibilities: $x_2 = 0$, and $x_1 = 2$. If we substitute $x_2 = 0$ into the first equation, we obtain $x_1 = 0$, which means that one of the equilibria is $x^e = [0; 0]$. Substituting $x_1 = 2$ transforms the first equation into

$$0 = -4 + 0.5x_2^2$$

which has solutions $x_2 = \sqrt{8}$ and $x_2 = -\sqrt{8}$. We can therefore conclude that the system has two more equilibria: $x^e = [2; \sqrt{8}]$ and $x^e = [2; -\sqrt{8}]$.

In our simulations, we will initially set p to zero, and solve the system on the interval $0 \leq t \leq 5$ for $x_0 = [1; 1]$, $x_0 = [3; -1]$, $x_0 = [-1; -1]$ and $x_0 = [-2; 2]$. We begin by creating the following .sci file that describes the right hand side of the equation

```
function  y = dem2(t, x, p)
   y(1)=-2*x(1)+0.5*(x(2)^2);
   y(2)=x(1)*x(2)-2*x(2)+p;
endfunction
```

Given values for t, x and p, this function outputs a *vector* with components $y(1)$ and $y(2)$. We then enter the following sequence of commands

```
p=0;
t0=0;
x0=[1;1];
t=0:0.01:5;
y=ode(x0,t0,t,list(dem2,p));
```

which produces the solution that originates in $x_0 = [1; 1]$. Note that

$$y = \left[\begin{array}{cccccc} x_1(0) & x_1(0.01) & x_1(0.02) & \dots & x_1(4.99) & x_1(5.00) \\ x_2(0) & x_2(0.01) & x_2(0.02) & \dots & x_2(4.99) & x_2(5.00) \end{array} \right]$$

is a matrix of dimension 2×501, whose entries represent the values that the solution takes at different points in time. To extract $x_1(t)$ from this matrix, type in

```
x1=y(1,:);
```

where (1,:) indicates that we are interested in row 1 and *all* the columns of y. Similarly, we can obtain $x_2(t)$ by entering

```
x2=y(2,:);
```

The sequence of commands

```
x0=[3;-1];
y=ode(x0,t0,t,list(dem2,p));
q1=y(1,:);
q2=y(2,:);
```

computes the solution that corresponds to $x_0 = [3; -1]$, and stores its components $x_1(t)$ and $x_2(t)$ as vectors q_1 and q_2, respectively. Proceeding in a similar manner, we can obtain the solutions for $x_0 = [-1; -1]$ and $x_0 = [-2; 2]$ as well (in the following, we will label them z_1, z_2 and w_1, w_2, respectively).

To plot component $x_1(t)$ for all four solutions, enter

```
plot(t,x1,t,q1,t,z1,t,w1)
```

which produces the graph shown in Fig. 1. Component $x_2(t)$ can be obtained using command

```
plot(t,x2,t,q2,t,z2,t,w2)
```

(the corresponding graph shown in Fig. 2). It is not difficult to see that equilibrium $x^e = [0; 0]$ is an *attractor* for this system, since both $x_1(t)$ and $x_2(t)$ approach zero as $t \to \infty$. The other two equilibria are "invisible", which suggests that they are *unstable.*

A somewhat different way to show that $x^e = [0; 0]$ is an attractor is to create a *phase plot*, in which state x_2 is shown as a function of x_1. We can create such a plot using the command

```
plot(x1,x2,1,1,'.')
```

which produces the graph shown in Fig 3. The terms 1,1,'.' in this expression display the initial condition $x_0 = [1; 1]$ as a small circle on the graph (which is a convenient way to indicate the starting point of the solution). Note that time does not appear explicitly in this plot, although it is used to construct each point of the curve - the first point corresponds to pair $(x_1(t_0), x_2(t_0))$, the second to $(x_1(t_1), x_2(t_1))$, and so on until $(x_1(t_f), x_2(t_f))$ (where t_f represents the final time point of the simulation).

The attractive nature of $x^e = [0; 0]$ is easily seen if we represent all four solutions on a single phase plot (together with their initial conditions). This can be done using command

```
plot(x1,x2,1,1,'.',q1,q2,3,-1,'.',z1,z2,-1,-1,'.',w1,w2,-2,2,'.',)
```

which gives rise to the graph shown in Fig. 4. This graph provides a nice geometric interpretation for the notion that $x^e = [0; 0]$ acts like a sort of "magnet" for solutions that originate from very different initial conditions. It is also interesting to note that the attractor in this case is a *point* in the phase plot. You will see in Project 2 that this is only one of several possibilities, and that attractors can sometimes have much more complex geometrical shapes.

It is instructive to see what changes in the system when the parameter value is set to $p = 2$. To examine this scenario, we will solve the differential equations for the same set of initial conditions as before, this time on the interval $0 \leq t \leq 10$. The first of these four solutions can be obtained by entering the following sequence of commands (the remaining three can be found in a similar manner)

```
p=2;
t0=0;
x0=[1;1];
t=0:0.01:10;
y=ode(x0,t0,t,list(dem2,p));
x1=y(1,:);
x2=y(2,:);
```

As in the previous case, we will first plot $x_1(t)$ and $x_2(t)$ for the four initial conditions, using commands

```
plot(t,x1,t,q1,t,z1,t,w1)
```

and

```
plot(t,x2,t,q2,t,z2,t,w2)
```

The corresponding graphs (which are shown in Figs. 5 and 6) indicate that all the solutions converge to a stable equilibrium, but this equilibrium is not $[0 \;\; 0]$ any more. To identify the location of this equilibrium more precisely, use commands

```
x1(1001)
```

and

```
x2(1001)
```

(note that y is a matrix of size $2 \times 1,001$ in this case). These commands produce 0.3825712 and 1.2367739, respectively, which tells us that the stable equilibrium for $p = 2$ is

$$x^e = \begin{bmatrix} 0.3825712 \\ 1.2367739 \end{bmatrix}$$

The phase plot for this system is shown in Fig. 7 (this plot was obtained in the same way as before). Geometrically, it is quite similar to the one in Fig. 4, the main difference being that the attractor is *not* located at the origin any more.

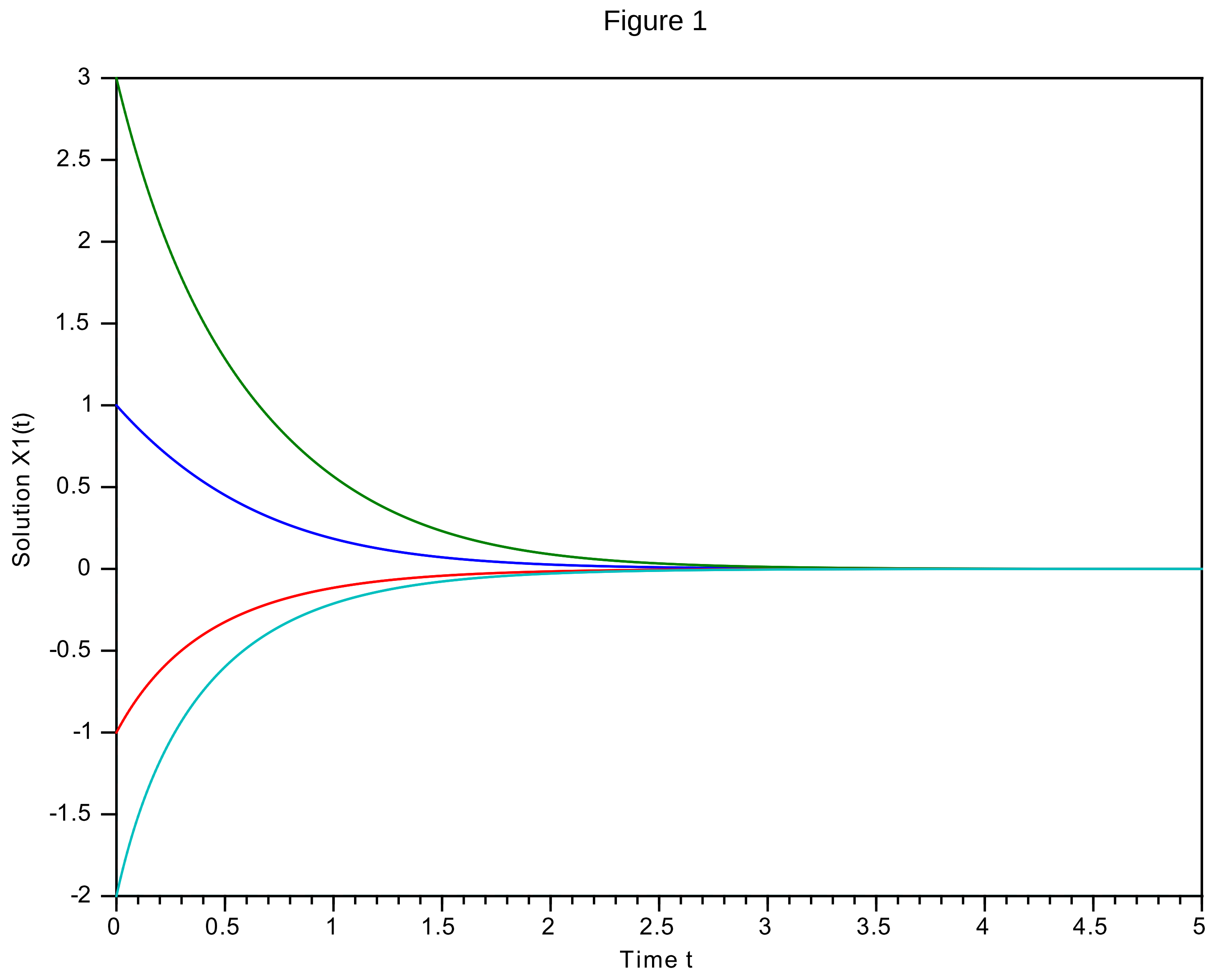


Figure 1
Solution X1(t)
Time t
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

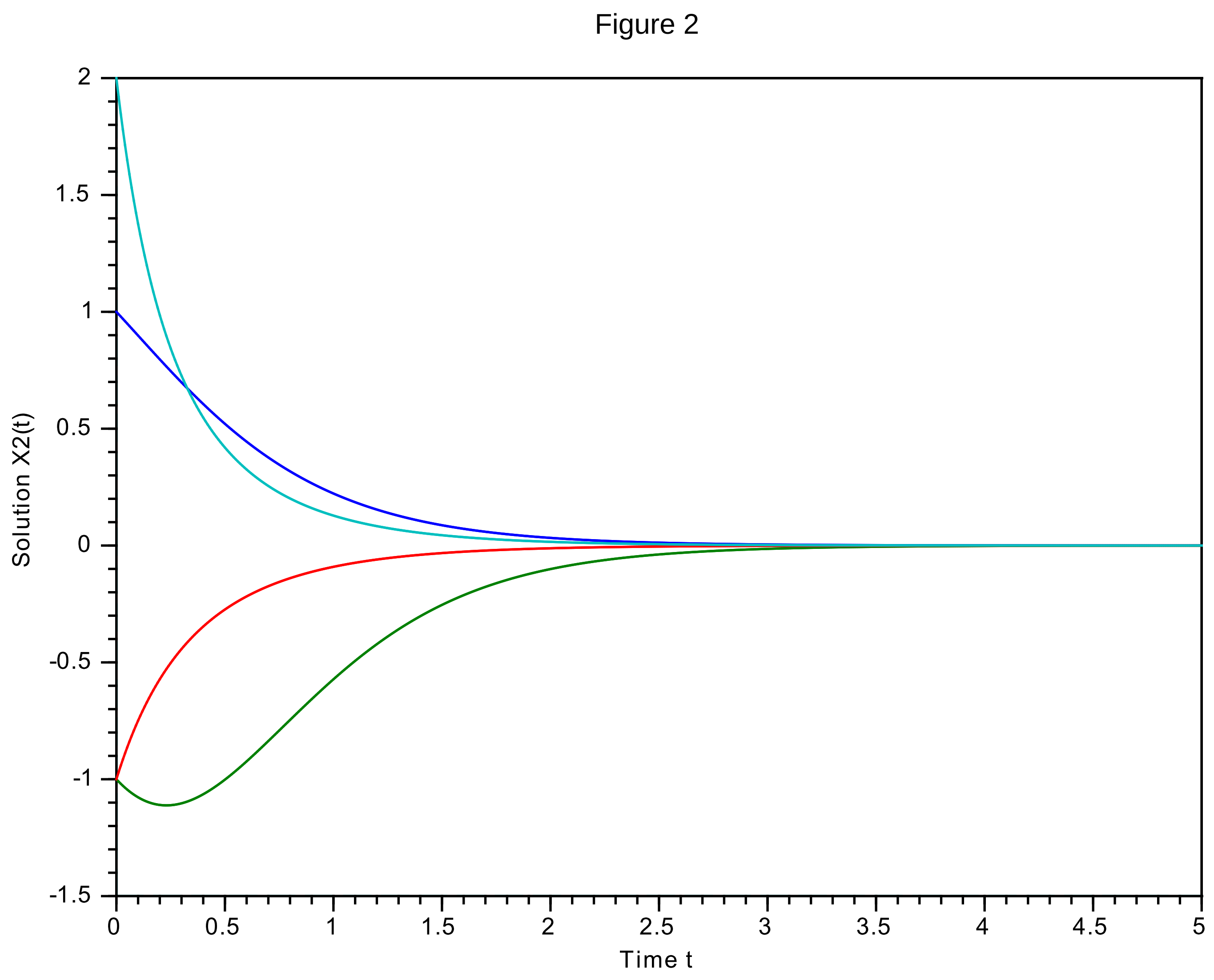


Figure 2
Solution X2(t)
Time t
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

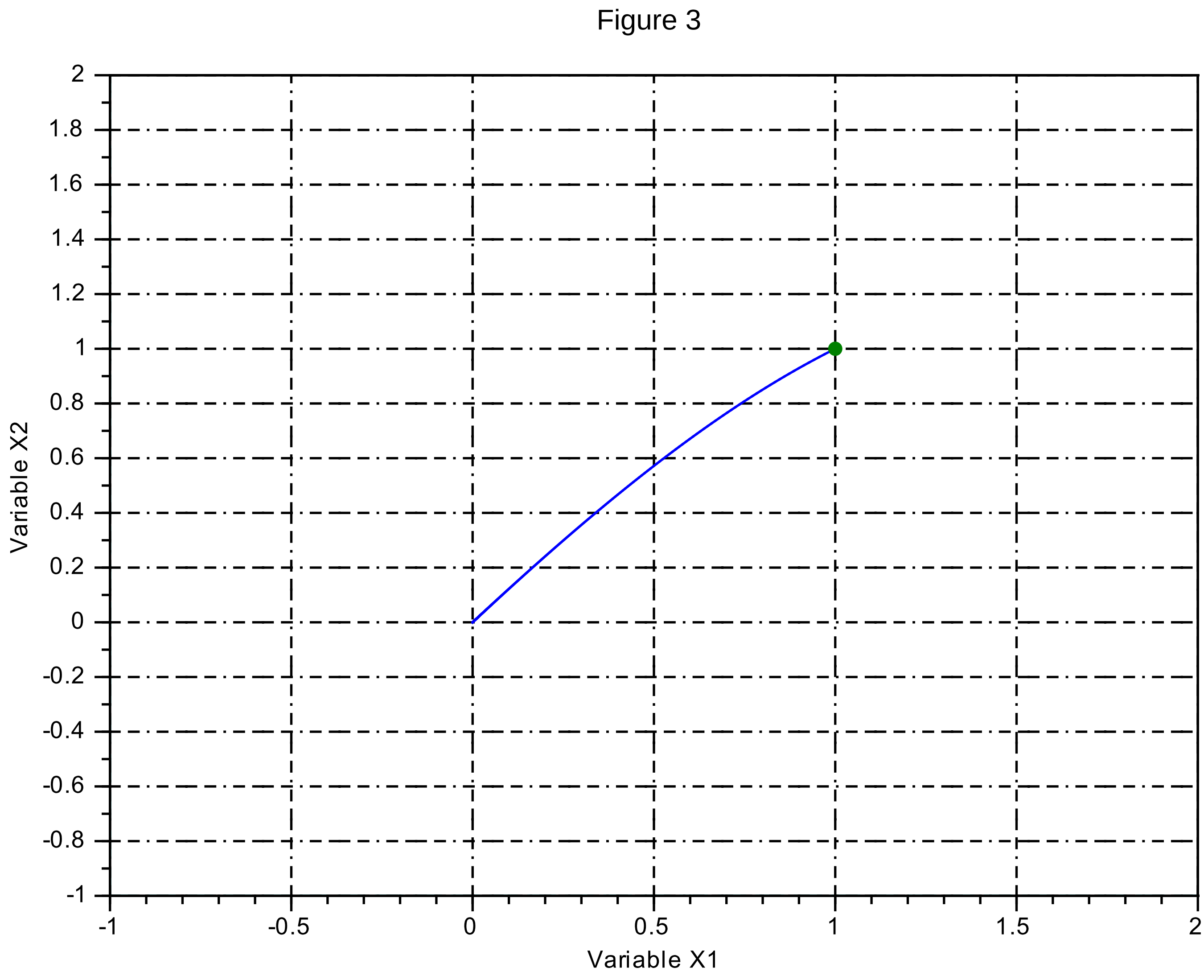


Figure 3
Variable X2
Variable X1
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1
1.5
2

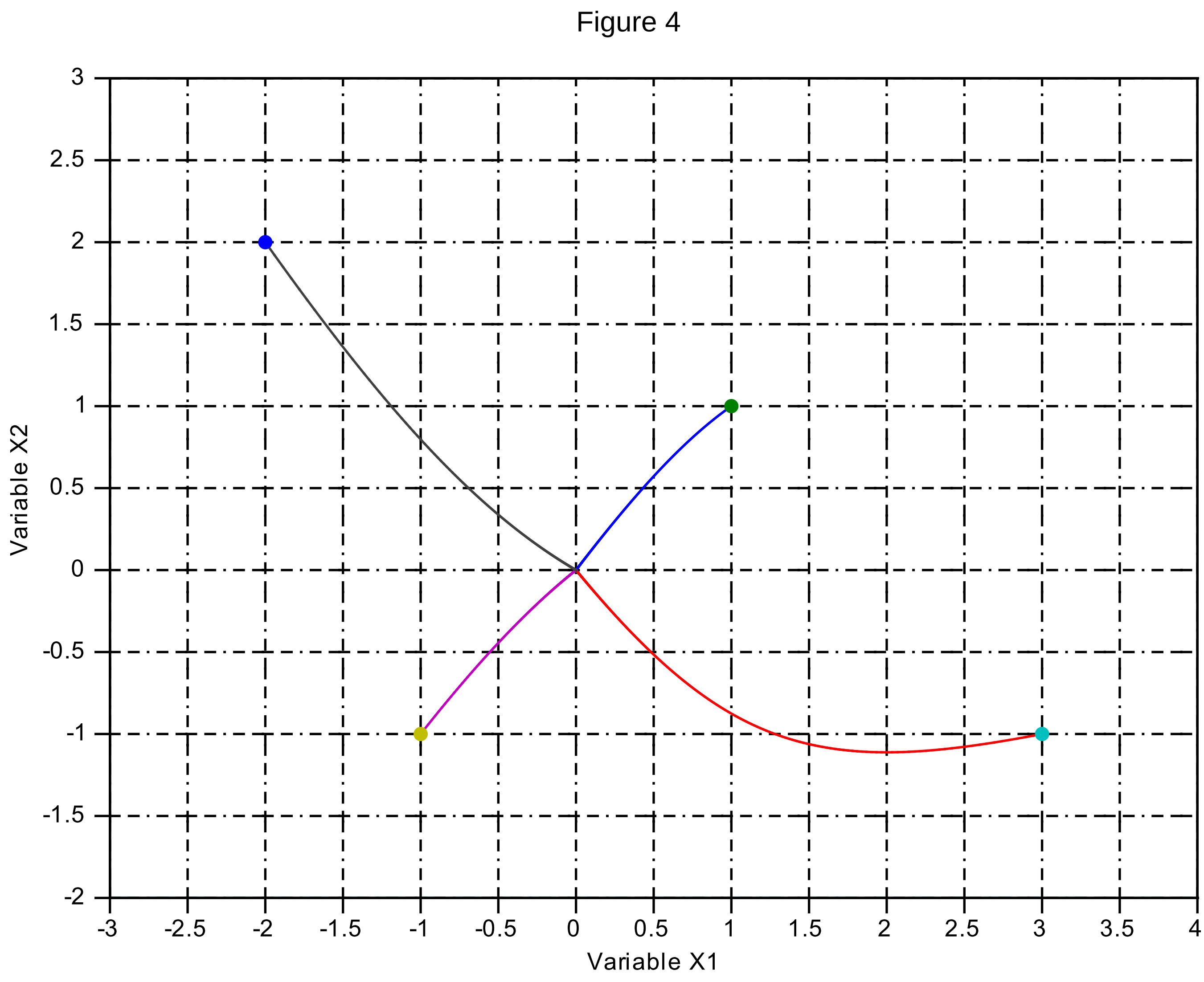


Figure 4
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
4
Variable X1
Variable X2

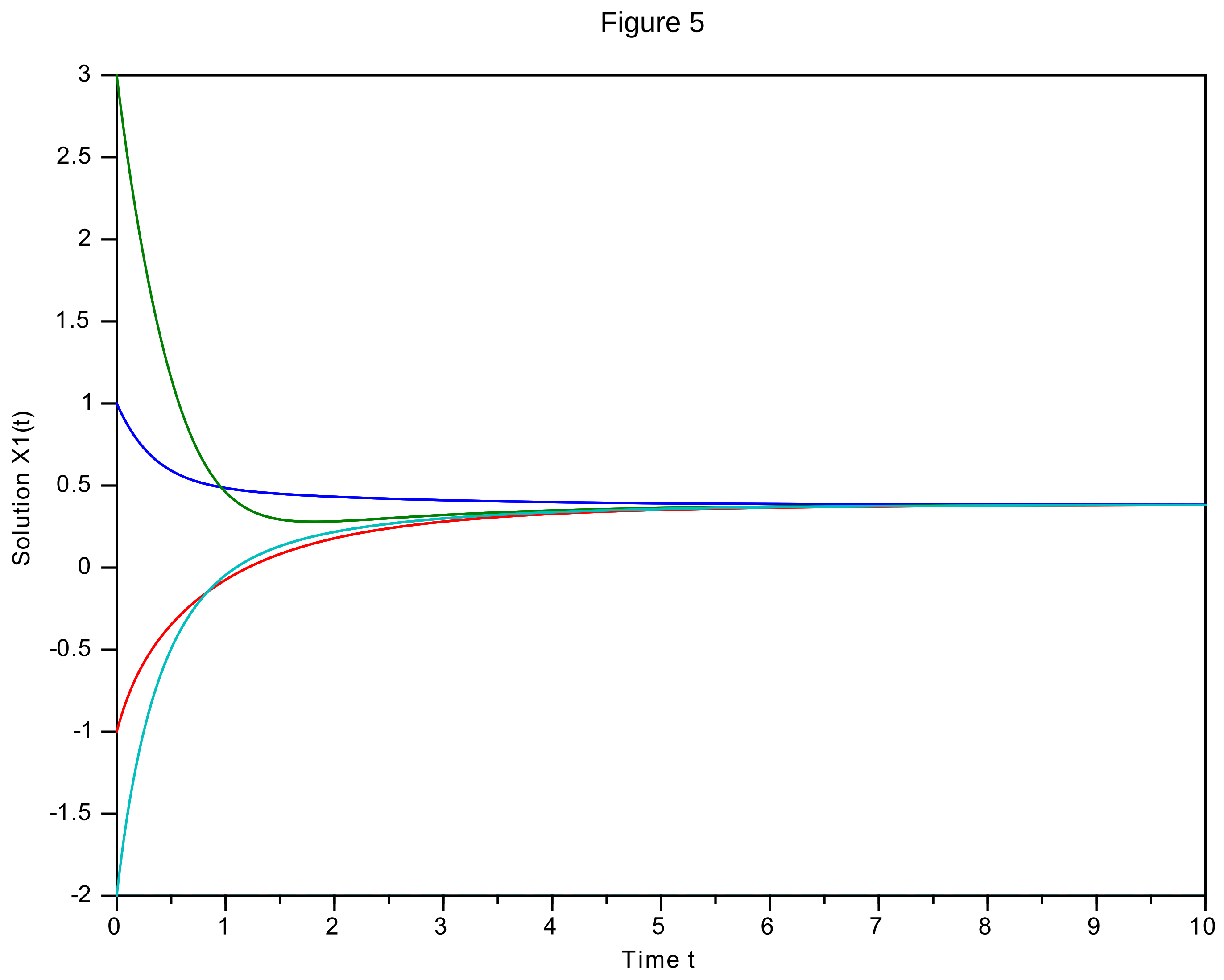
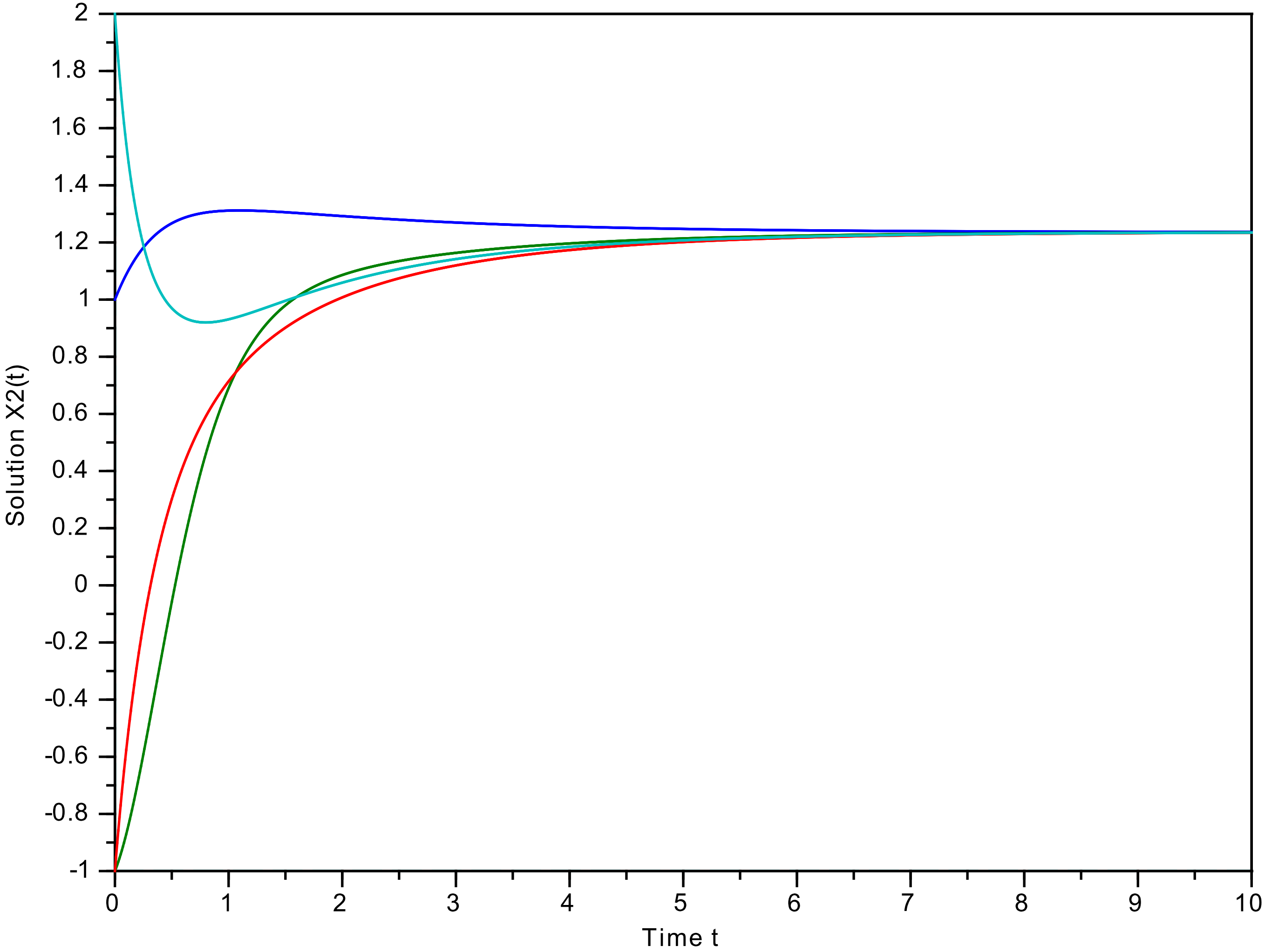


Figure 5
Solution X1(t)
Time t
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
1
2
3
4
5
6
7
8
9
10

Figure 6



2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
Solution X2(t)
0
1
2
3
4
5
6
7
8
9
10
Time t

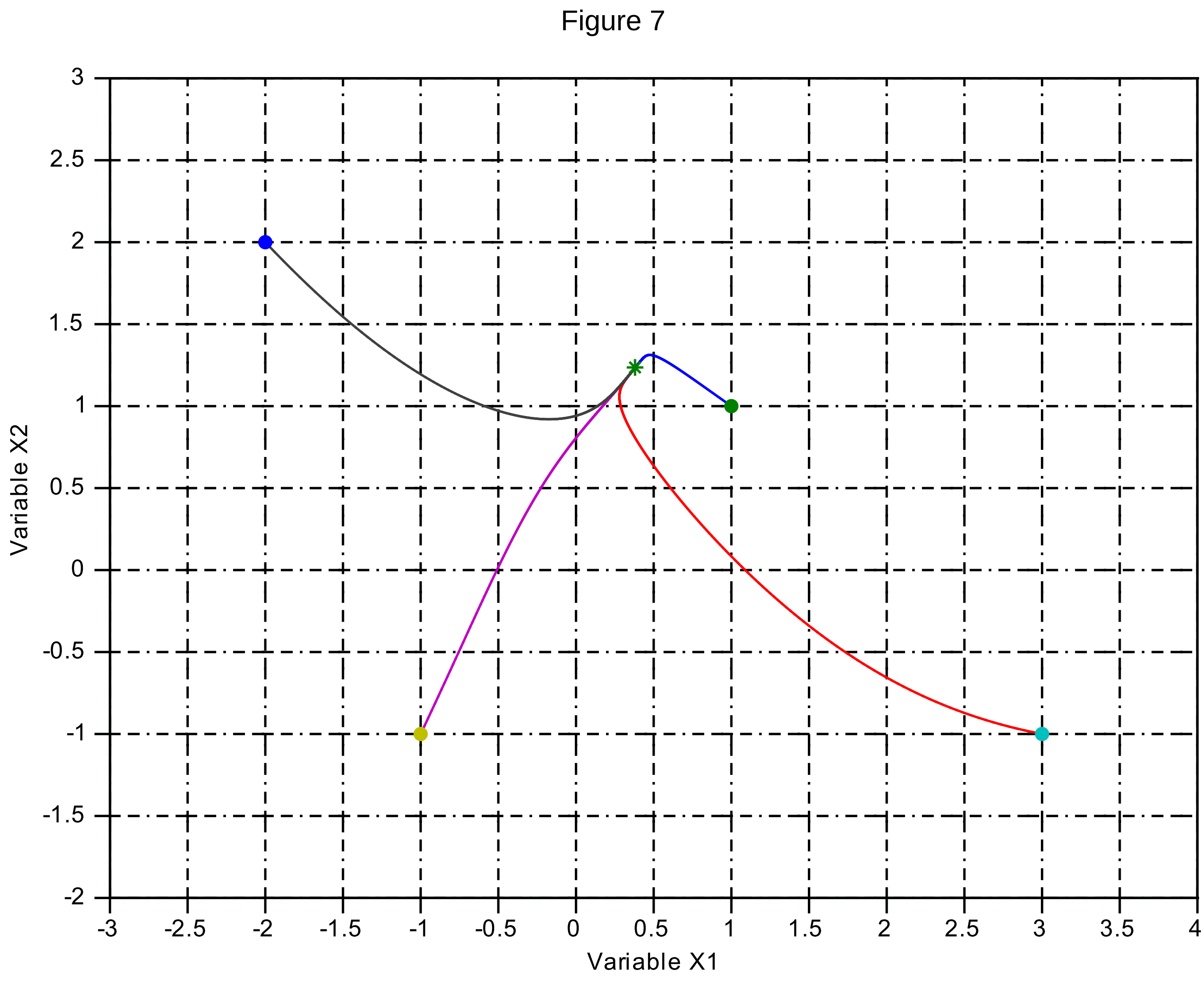


Figure 7
Variable X2
Variable X1
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
4