MODERN NETWORK ANALYSIS I

LECTURE NOTES

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GAUSSIAN ELIMINATION

Elementary transformations

Type 1 Divide a row by a number.

Type 2 Multiply a row by a number and add it to another row.

Type 3 Interchange two rows.

The process of Gaussian elimination can always be represented as a sequence of elementary transformations. In some cases, only transformations of Type 2 are necessary.

Performing *any* elementary transformation on a matrix is equivalent to multiplying it on the left with a special matrix. For the three types of transformations, these special matrices are of the following form

Type 1

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiplying a 3×3 matrix A on the left by P_1 corresponds to multiplying its second row by a.

Type 2

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 0 & 1 \end{bmatrix}$$

Multiplying a 3×3 matrix A on the left by P_2 corresponds to multiplying its first row by b and adding it to the third row. Note that the determinant of P_2 is equal to 1.

Type 3

$$P_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Multiplying a 3×3 matrix A on the left by P_3 corresponds to interchanging the first and third rows.

Some applications of Gaussian elimination

1) Solving linear equations $A\,x = b$.

2) Computing the determinant of a matrix.

3) Determining the rank of a matrix.

4) Matrix inversion.

EXAMPLE 1

Consider the following matrix

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

The Gaussian elimination proceeds as follows :

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow P \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A \Rightarrow A_1 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & -5 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P \cdot A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \cdot A_1 \quad \Rightarrow \quad A_2 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & -5 \\ 0 & 1 & -2 \end{bmatrix}$$

$$P \cdot A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \cdot A_2 \quad \Rightarrow \quad A_3 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & -5 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Note that here we used only Type 2 transformations, so we can compute the determinant directly.

Matrix inversion

We now show how to invert the matrix in Example 1. We first form an *augmented* matrix made up of matrix A and the identity matrix

$$\begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 2 & -5 & -2 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 2 & -5 & -2 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 2 & -5 & -2 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 2 & -5 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 2 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 2 & 0 & -2 & -4 & 10 \\ 0 & 0 & 1 & 0 & -1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 3 & -6 \\ 0 & 2 & 0 & -2 & -4 & 10 \\ 0 & 0 & 1 & 0 & -1 & 2 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 3 & -6 \\ 0 & 1 & 0 & -1 & -2 & 5 \\ 0 & 0 & 1 & 0 & -1 & 2 \end{bmatrix}$$

Rank determination

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 2 \\ 0 & 4 & 2 & 0 \end{bmatrix}$$

After Gaussian elimination, we obtain

$$A^* = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It is now obvious that rank(A) = 2, since the largest nonsingular submatrix of A* has dimension 2 × 2.

SINGULAR SYSTEMS OF EQUATIONS

Consider the matrix A from the previous example, and solve $A\ x = b$ in three different cases.

Case 1 (when the right hand side is zero)

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 2 \\ 0 & 4 & 2 & 0 \end{bmatrix} x = 0 \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solutions :

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}s - t \\ -\frac{1}{2}s \\ s \\ t \end{bmatrix} = s \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Here there are infinitely many solutions, since t and s are arbitrary real numbers.

Case 2

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 2 \\ 0 & 4 & 2 & 0 \end{bmatrix} x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}$$

This is an *inconsistent* system of equations, so there are *no* solutions.

Case 3

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 2 \\ 0 & 4 & 2 & 0 \end{bmatrix} x = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

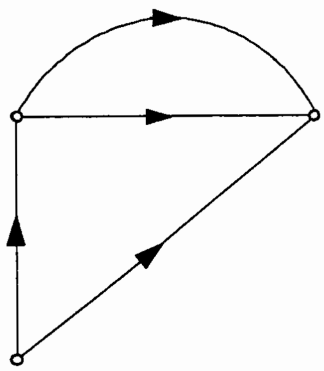
The system is consistent, with infinitely many solutions

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}s - t + 1 \\ -\frac{1}{2}s \\ s \\ t \end{bmatrix} = s \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

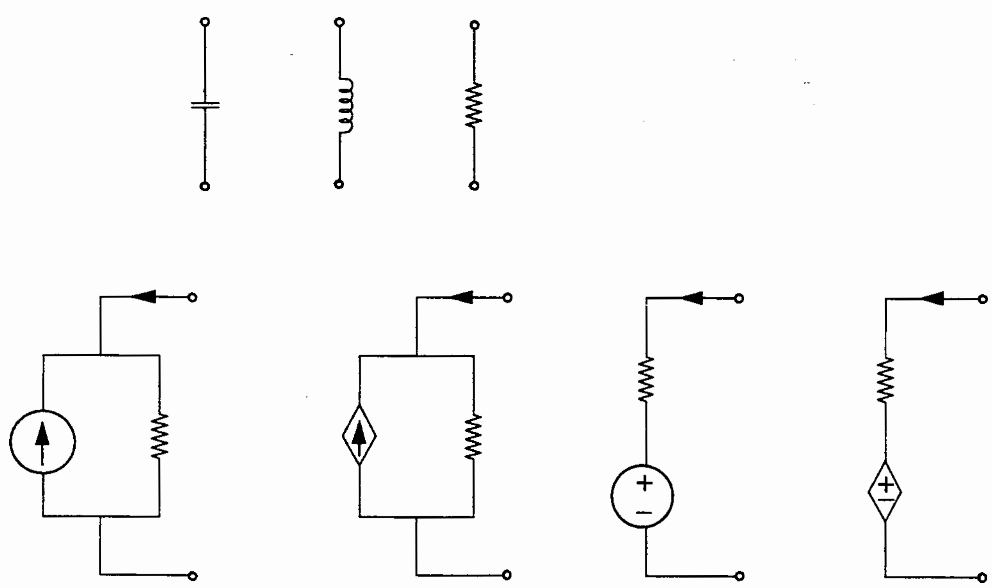
GRAPHS

GRAPHS

A graph can be thought of as an entity made up of v vertices (nodes) and e edges (branches) that connect them. If directions are specified for each edge, we have a *directed graph.*

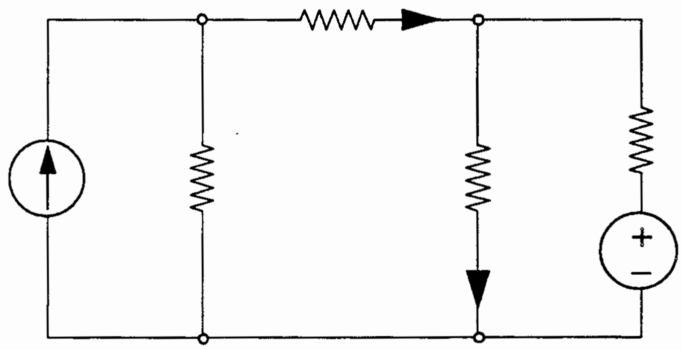


Directed graphs can be used to describe circuits. In this process, one edge can represent *any* of the following :

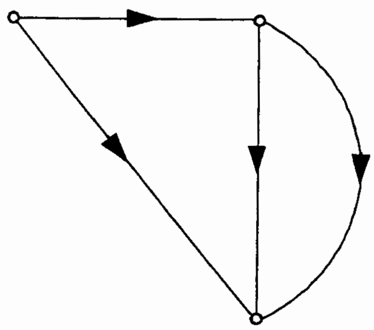


Note that in the case of sources the direction of the corresponding graph edge is *preassigned.*

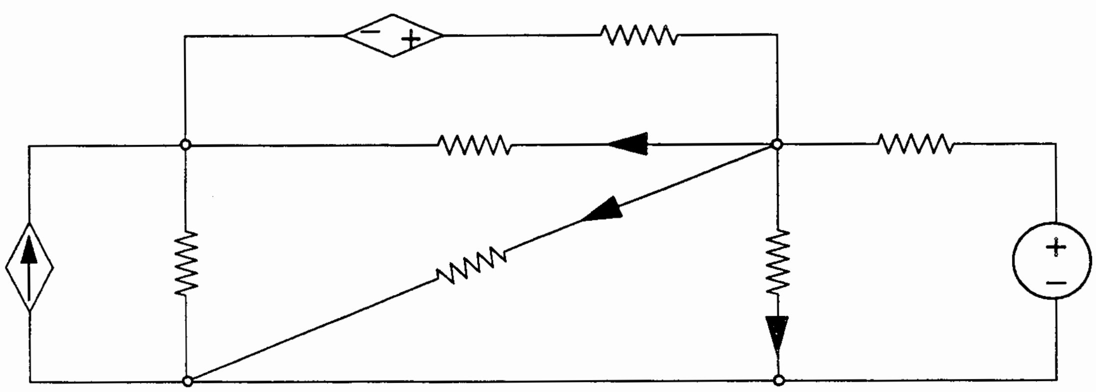
EXAMPLE 1



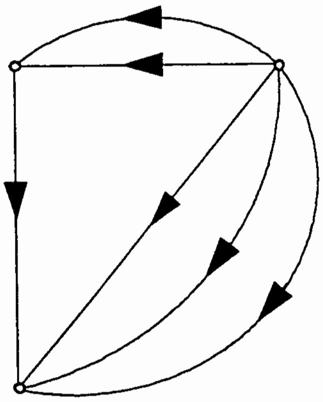
Graph



EXAMPLE 2



Graph



SOME DEFINITIONS

SUBGRAPHS

A subgraph is any subset of vertices and edges from the original graph. Note that a single vertex is a legitimate subgraph.

DEGREE OF A VERTEX

The degree of a vertex is equal to the number of edges that are connected to it. It is usually denoted by deg(v).

LOOPS

A loop is a subgraph in which all vertices have degree = 2.

PATHS

A path is a subgraph in which all vertices have degree = 2 except *two*, which have degree = 1.

CONNECTED GRAPHS

A connected graph is a graph in which there is a path between any two vertices.

SIMPLE GRAPHS

A simple graph is a graph with *no* multiple edges and *no* self-loops.

TREES

A tree is a connnected subgraph that contains *all* v vertices of the original graph and has *no* loops.

A tree can be determined by disconnecting edges of the original graph one at a time, until no more loops are left. It can be shown that in a graph with v vertices a tree has *exactly* $v - 1$ edges. Edges that are not in the tree are referred to as *links*, and there are $e - v + 1$ of them.

Number of candidates for a tree

The number of candidates for a tree is equivalent to the number of different ways for selecting v - 1 out of a total of e edges. This number is

$$\frac{e\,!}{(e - v + 1)\,!\,(v - 1)\,!}$$

Actual number of different trees

Not all tree candidates are legitimate trees, since some choices of v - 1 edges can contain loops. In order to determine the exact number of trees, it is first necessary to form an auxiliary matrix H using the following set of rules :

1) Select a reference node. Only the remaining v - 1 nodes are considered in forming H.

2) H is a (v - 1) × (v - 1) matrix, with elements

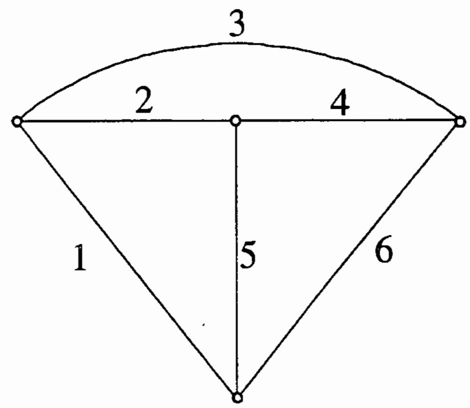
$$h_{ii} = \deg(v_i)$$

$$h_{ij} = -\,\mathrm{K}(i, j)$$

where K(i, j) represents the number of edges connecting nodes i and j (K(i, j) = 0 when nodes i and j are *not* connected). Note that in a simple graph K(i, j) is either 0 or 1.

3) The actual number of trees is equal to det (H).

EXAMPLE 3



There are 20 candidates for a tree

1 2 3	2 3 4	3 4 5	4 5 6
1 2 4	2 3 5	3 4 6	
1 2 5	2 3 6	3 5 6	
1 2 6	2 4 5		
1 3 4	2 4 6		
1 3 5	2 5 6		
1 3 6			
1 4 5			
1 4 6			
1 5 6			

The auxiliary matrix H is

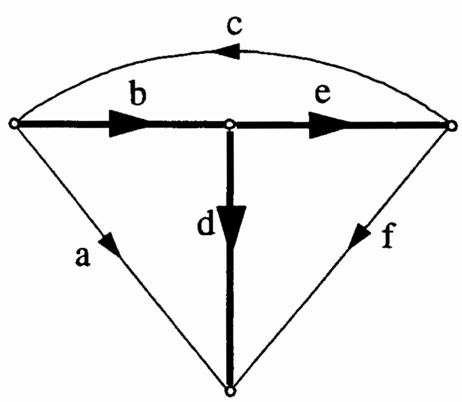
$$H = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

Since det $(H) = 16$, it follows that exactly 4 of the candidates are not legitimate. These are {1 2 5}, {1 3 6}, {2 3 4} and {4 5 6}.

FUNDAMENTAL CUTSETS

A cutset is a set of edges whose removal breaks up the original graph into exactly two disconnected subraphs. A *fundamental cutset* is a special kind of cutset which is always defined with respect to a tree, and contains exactly one tree branch.

EXAMPLE 4

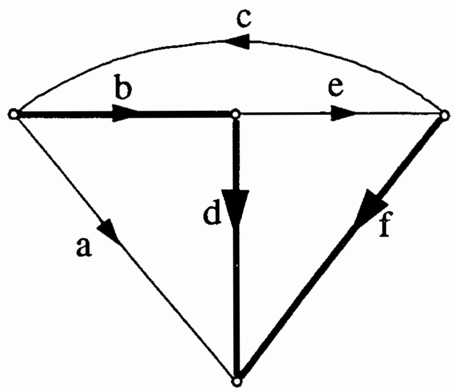


{a, ***b***, c}

{a, ***d***, f}

{f, ***e***, c}

For a different choice of tree



{e, *f*, c}

{a, *d*, e, c}

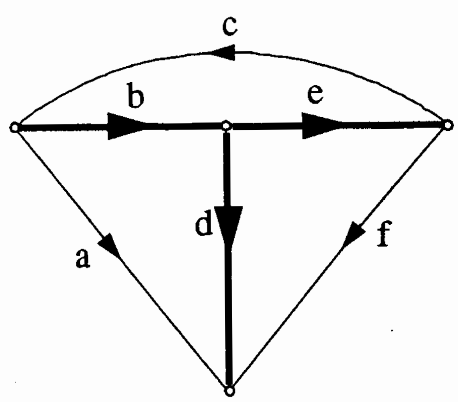
{a, *b*, c}

Fundamental cutsets can be used to systematically formulate *Kirchoff's current law equations.* The definition of a fundamental cutset secures that all such equations are *independent.* In this context, it is also convenient to introduce the *orientation* of a cutset, which coincides with the orientation of the tree branch which defines it.

FUNDAMENTAL LOOPS

A *fundamental loop* is a special kind of loop which is always defined with respect to a tree, and contains exactly one link. All fundamental loops can be obtained by adding links to the tree one at a time.

EXAMPLE 5

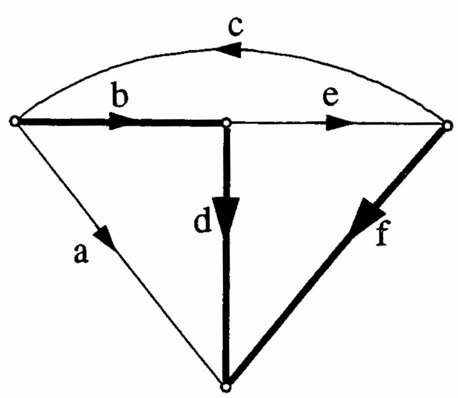


{*f*, d, e}

{***c***, b, e}

{***a***, d, b}

For a different choice of tree



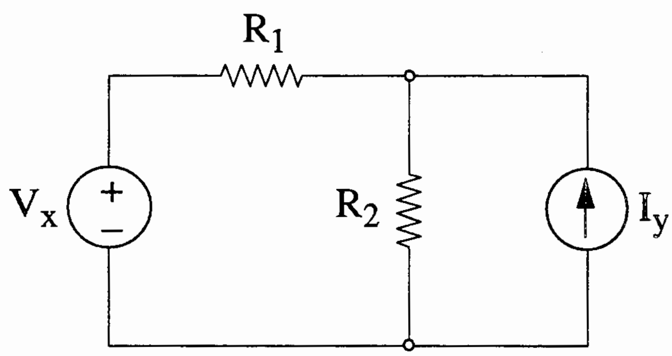
{***c***, b, d, f}

{***e***, f, d}

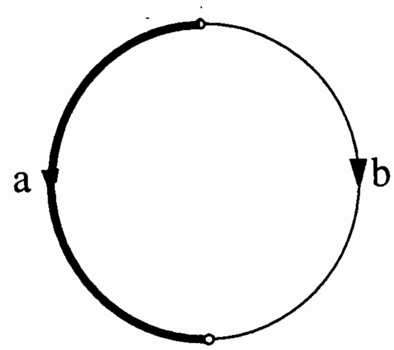
{***a***, d, b}

Fundamental loops can be used to systematically formulate *Kirchoff's voltage law* equations, which are guaranteed to be independent. Again, it will be convenient to introduce the *orientation* of a loop, which coincides with the orientation of the link which defines it.

EXAMPLE 6



Graph



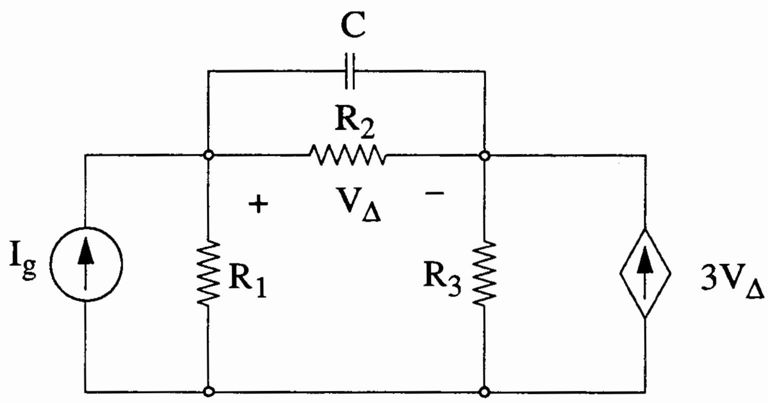
KCL equations

$$i_a + i_b = 0$$

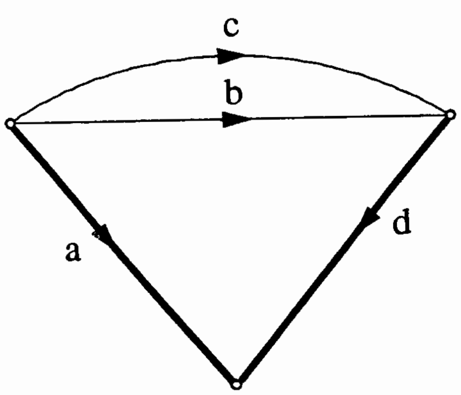
KVL equations

$$v_a - v_b = 0$$

EXAMPLE 7



Graph



KCL equations

$$i_a + i_b + i_c = 0$$

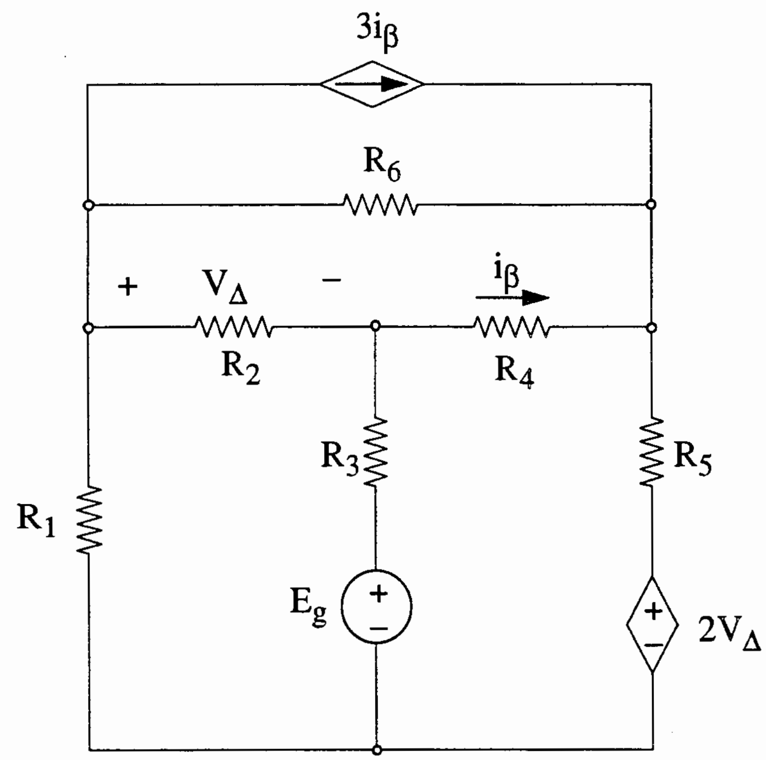
$$i_d - i_b - i_c = 0$$

KVL equations

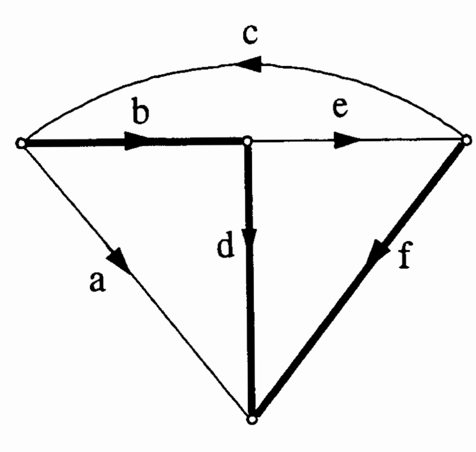
$$v_a - v_d - v_b = 0$$

$$v_a - v_d - v_c = 0$$

EXAMPLE 8



Graph



KCL equations

$$i_a + i_b - i_c = 0$$

$$i_a + i_d + i_e - i_c = 0$$

$$i_f - i_e + i_c = 0$$

KVL equations

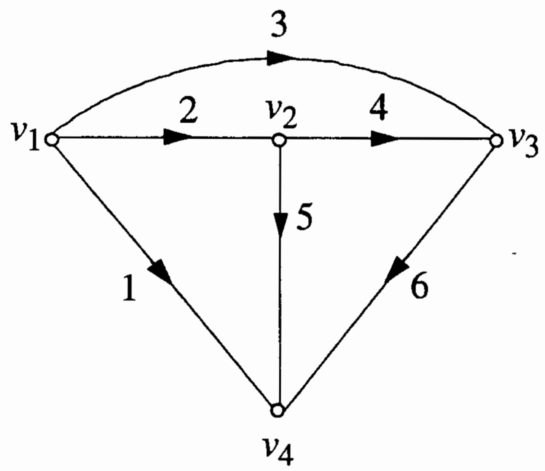
$$v_a - v_d - v_b = 0$$

$$v_c - v_f + v_d + v_b = 0$$

$$v_e + v_f - v_d = 0$$

NETWORK MATRICES

Graph **G** (*without a specified tree*)



Rules for constructing an incidence matrix

a) Select a *reference node*, and consider only the v - 1 remaining vertices (the dimension is $(v - 1) \times e$).

b) For a branch e_k : + 1 if its orientation is *out* of the vertex
- 1 if its orientation is *into* the vertex

For the graph shown above, the incidence matrix is :

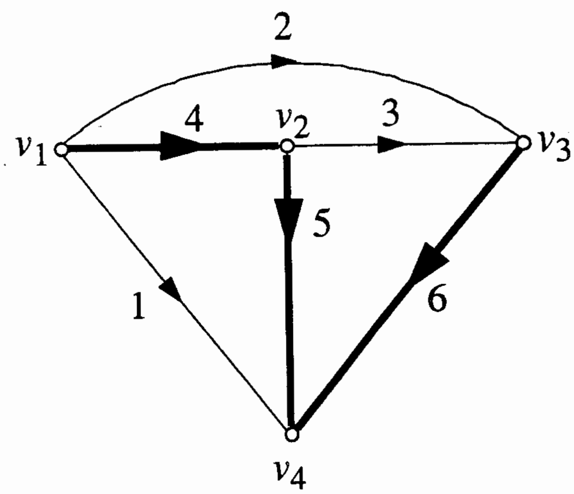
$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{bmatrix}$$

What happens if we do not disregard the reference node ? We would have a $v \times e$ matrix A_a :

$$A_a = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 & -1 & -1 \end{bmatrix} \end{array}$$

This matrix has rank $< v$! It can be shown that for *any* connected graph rank(A) = v - 1 (that is, A has *full rank*).

Graph **G** (*with a specified tree*)



IMPORTANT NOTE : When a tree is specified, always number the *links first* and *tree branches last.*

Renumbered incidence matrix :

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

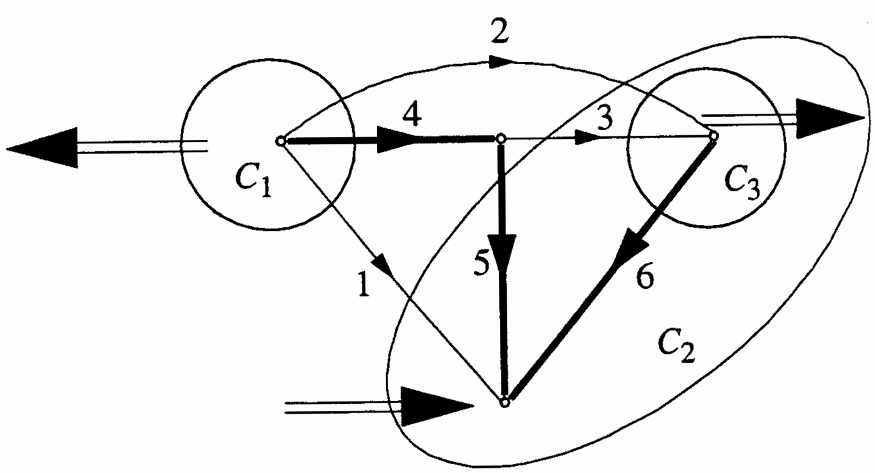
THE FUNDAMENTAL CUTSET MATRIX

Rules for construction

1) Identify the fundamental cutsets and assign an orientation to each one (this orientation is defined by the corresponding tree branch).

2) For a branch e_k : + 1 if it agrees with the cutset orientation.
- 1 if it is opposite to the cutset orientation.

Fundamental cutsets for **G**



*Matrix Q_f for **G***

$$Q_f = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \end{array} \begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \end{bmatrix} \qquad \textbf{(4)}$$

General structure

$$Q_f = \begin{bmatrix} Q_{f11} & I \end{bmatrix}$$

There will always be an identity matrix in the part of Q_f corresponding to tree branches, *provided* that links are numbered first. Otherwise, this property *will not* hold in general.

THE FUNDAMENTAL LOOP MATRIX

Rules for construction

1) Identify the fundamental loops and assign an orientation to each one (this orientation is defined by the corresponding link).

2) For a branch e_k : + 1 if it agrees with the loop orientation.
- 1 if it is opposite to the loop orientation.

*Matrix B_f for **G***

$$B_f = \begin{array}{c} \\ l_1 \\ l_2 \\ l_3 \end{array} \begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix}$$

General structure

$$B_f = \begin{bmatrix} I & B_{f12} \end{bmatrix}$$

There will always be an identity matrix in the part of B_f corresponding to the links, provided that links are numbered first.

RELATIONSHIP BETWEEN Q_f AND B_f

The basic relationship

$$Q_f \cdot B_f^T = 0$$

Corrolary

$$Q_{f11} = -B_{f12}^T$$

RELATIONSHIP BETWEEN A AND B_f

The basic relationship

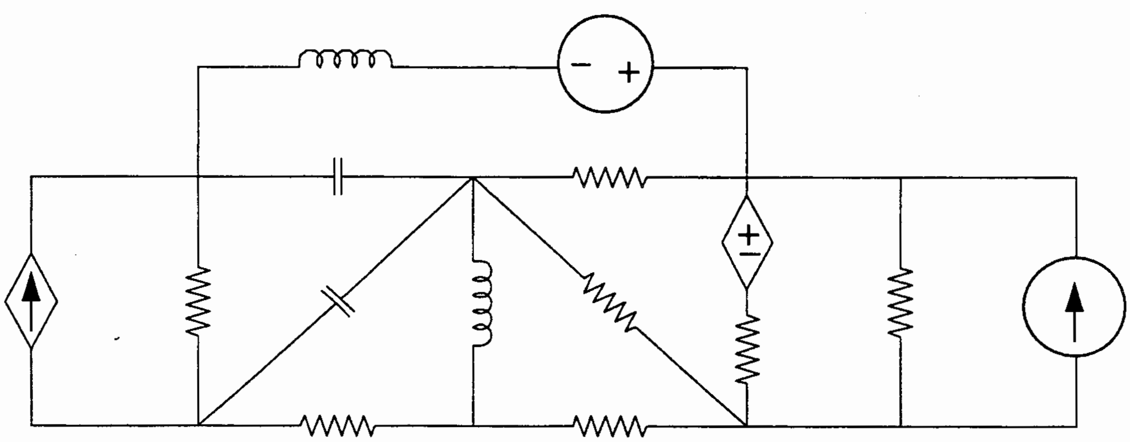
$$A \cdot B_f^{\,T} = 0$$

Corrolary

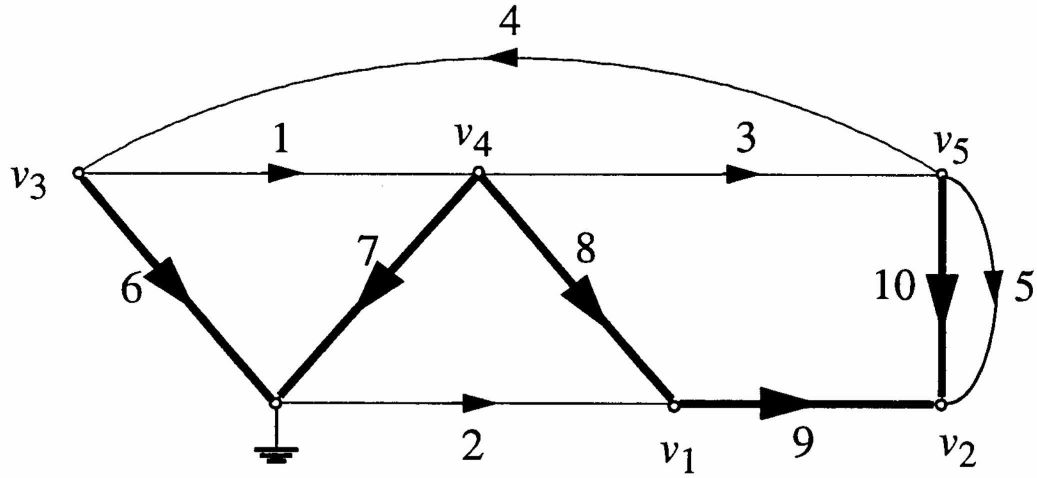
$$B_{f12}^{T} = -A_{12}^{-1} A_{11}$$

Note that matrix A_{12} is always nonsingular, since it is the incidence matrix of the tree.

EXAMPLE



Graph



Incidence matrix

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} \end{array} \\ \left[\begin{array}{cccccccccc} 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 \\ 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

Fundamental cutset matrix

$$
Q_f = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{array}
\begin{array}{c} \begin{array}{cccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} \end{array} \\
\left[\begin{array}{cccccccccc}
1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 1
\end{array}\right] \end{array}
$$

Fundamental loop matrix

$$
B_f = \begin{array}{c} \\ l_1 \\ l_2 \\ l_3 \\ l_4 \\ l_5 \end{array}
\begin{array}{c} \begin{array}{cccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} \end{array} \\
\left[\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & -1 & 1 & 1 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1
\end{array}\right] \end{array}
$$

CURRENT- VOLTAGE RELATIONSHIPS

Kirchoff's laws in the time domain can be written in matrix form

$$B_f v_e(t) = 0 \qquad (KVL) \qquad v_e(t) \equiv \begin{bmatrix} v_l(t) \\ ---- \\ v_t(t) \end{bmatrix}$$

$$Q_f\, i_e(t) = 0 \qquad (KCL) \qquad i_e(t) \equiv \begin{bmatrix} i_l(t) \\ ---- \\ i_t(t) \end{bmatrix}$$

There is a total of e equations (e - v + 1 equations by KVL and v - 1 equations by KCL). On the other hand, there are 2e unknowns (e currents and e voltages). We therefore need another e equations, which we obtain from the current-voltage relationships (CVR).

To make these relationships *algebraic*, we first need to write Kirchoff's laws in the Laplace transform

$$B_f V_e(s) = 0 \qquad ; \qquad Q_f I_e(s) = 0$$

RESISTORS

$$v_i(t) = R\, i_i(t) \quad \Rightarrow \quad V_i(s) = R\, I_i(s)$$

INDUCTORS

$$v_i(t) = L \frac{di_i}{dt} \quad \Rightarrow \quad V_i(s) = s L I_i(s) - L i_i(0)$$

CAPACITORS

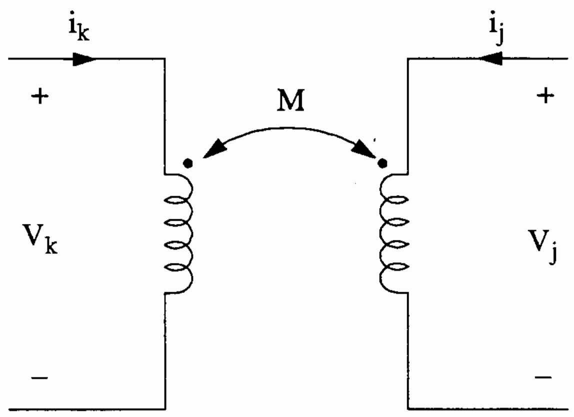
$$i_i(t) = C \frac{dv_i}{dt}$$

Therefore,

$$I_i(s) = s C V_i(s) - C v_i(0) \quad \Rightarrow \quad V_i(s) = \frac{1}{sC} I_i(s) + \frac{v_i(0)}{s}$$

MUTUAL INDUCTANCE

Case 1



$$v_k(t) = L_k \frac{di_k}{dt} + M \frac{di_j}{dt}$$

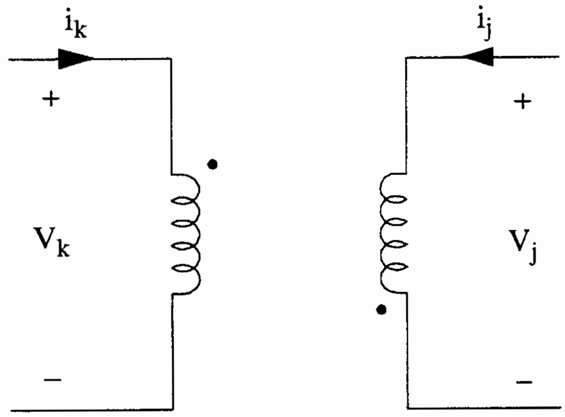
$$v_j(t) = L_j \frac{di_j}{dt} + M \frac{di_k}{dt}$$

In Laplace transform,

$$V_k(s) = sL_k I_k(s) - L_k i_k(0) + sMI_j(s) - Mi_j(0)$$

$$V_j(s) = sL_j I_j(s) - L_j i_j(0) + sMI_k(s) - Mi_k(0)$$

Case 2



$$v_k(t) = L_k \frac{di_k}{dt} - M \frac{di_j}{dt}$$

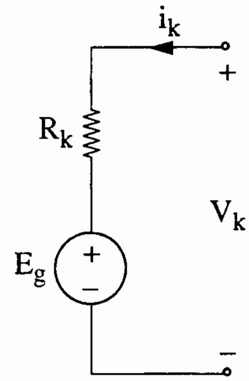
$$v_j(t) = L_j \frac{di_j}{dt} - M \frac{di_k}{dt}$$

In Laplace transform,

$$V_k(s) = sL_k I_k(s) - L_k i_k(0) - sMI_j(s) + Mi_j(0)$$

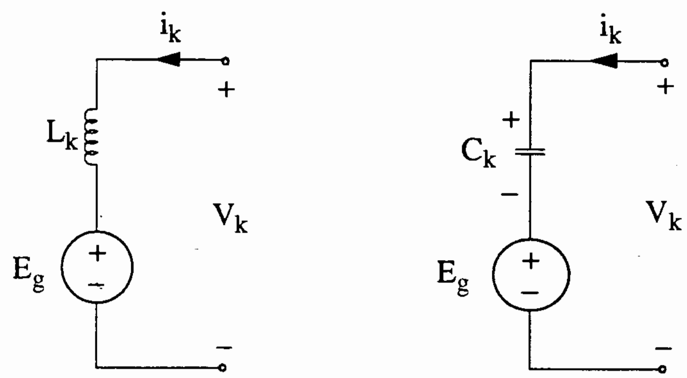
$$V_j(s) = sL_j I_j(s) - L_j i_j(0) - sMI_k(s) + Mi_k(0)$$

INDEPENDENT VOLTAGE SOURCES



$$V_k(s) = R_k I_k(s) + E_g(s)$$

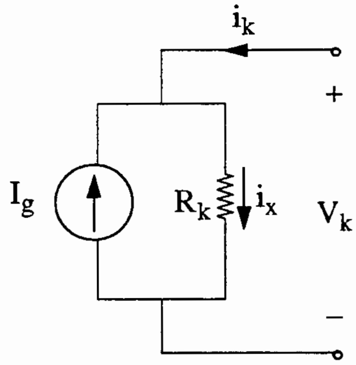
Variants with inductors and capacitors



$$V_k(s) = sL_k I_k(s) - L_k i_k(0) + E_g(s)$$

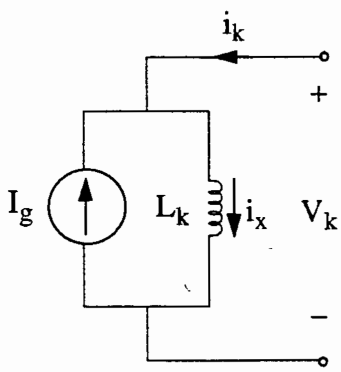
$$V_k(s) = \frac{1}{sC_k} I_k(s) + \frac{1}{s}\left(v_k(0) - e_g(0)\right) + E_g(s)$$

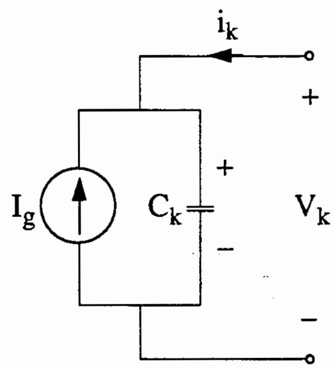
INDEPENDENT CURRENT SOURCES



$$V_k(s) = R_k I_k(s) + R_k I_g(s)$$

Variants with inductors and capacitors





$$V_k(s) = sL_k I_k(s) - L_k(i_k(0) + i_g(0)) + sL_k I_g(s)$$

$$V_k(s) = \frac{1}{sC_k} I_k(s) + \frac{1}{s} v_k(0) + \frac{1}{sC_k} I_g(s)$$

GENERAL FORMULATION OF CVR

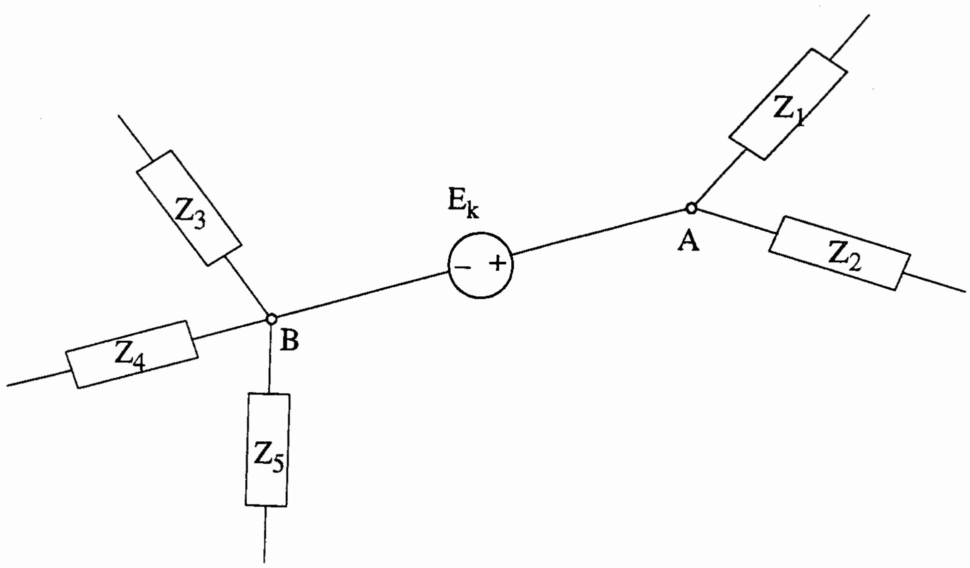
$$V_e(s) = Z_e(s) I_e(s) + E_e(s) - L_e i_e(0) + \frac{1}{s} v_e(0)$$

This general format covers all the elements considered so far. Matrix $Z_e(s)$ in the above equation is known as the *edge impedance matrix.*

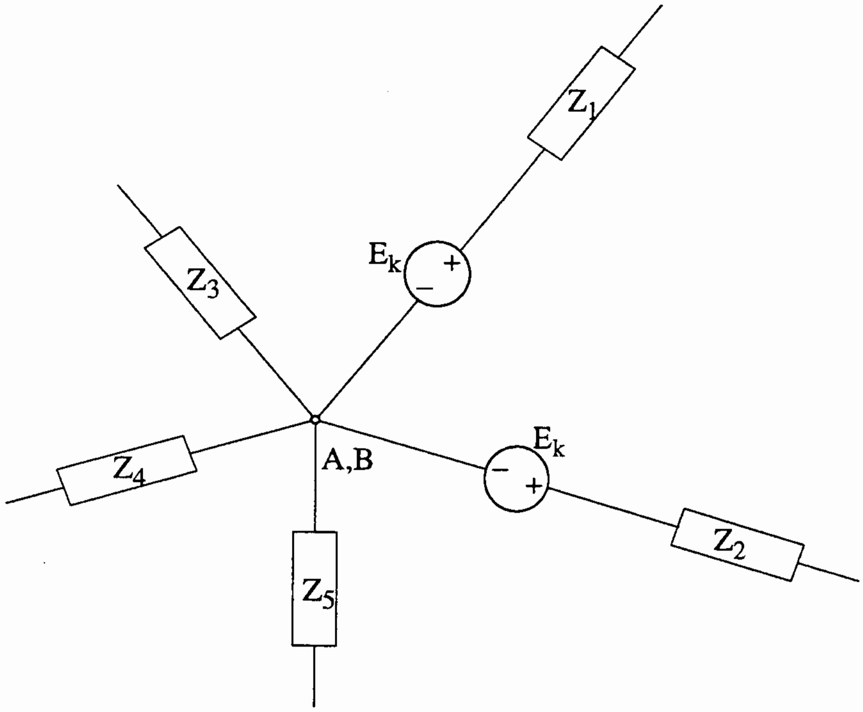
VOLTAGE SOURCE TRANSFORMATIONS

These transformations are necessary when there are no elements in series with an independent voltage source.

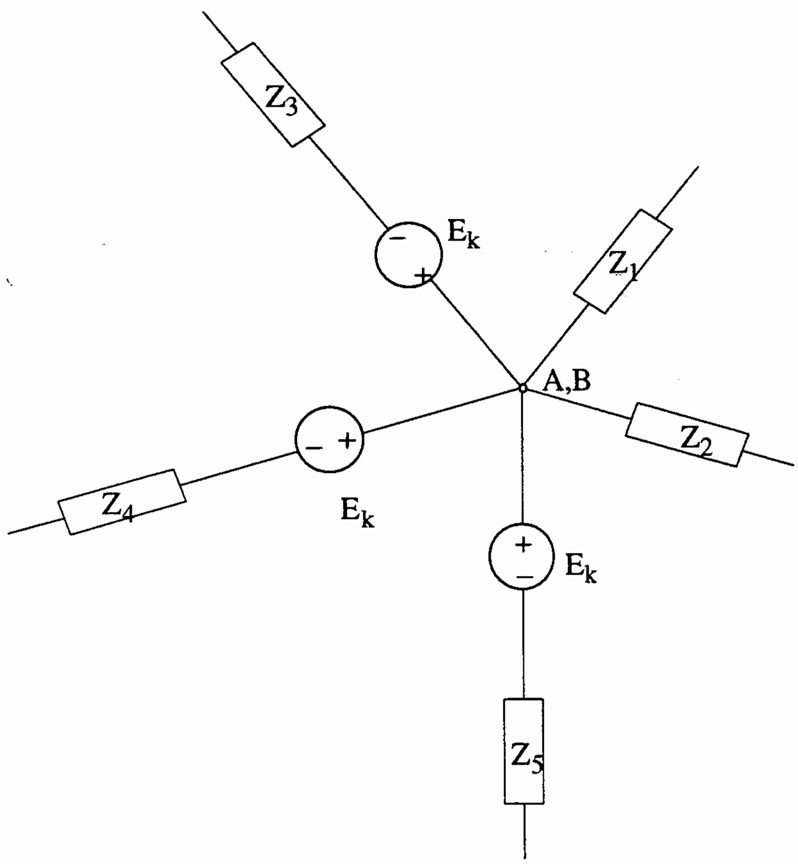
Generic situation



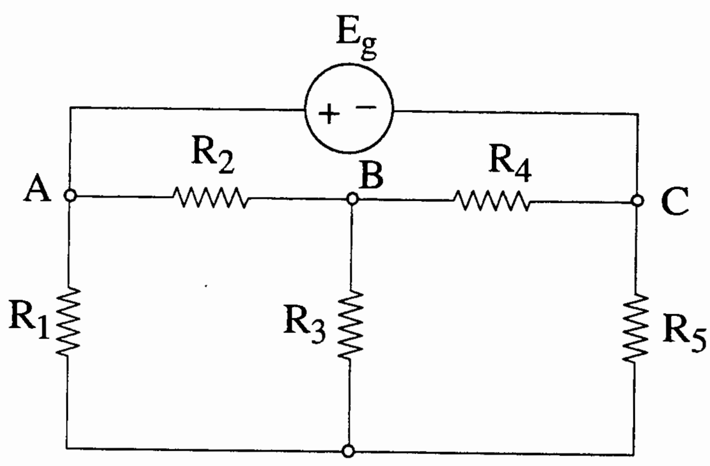
Equivalent representations



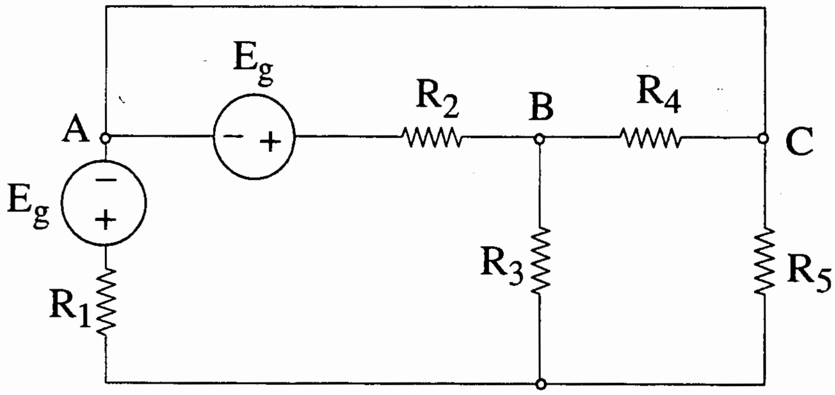
or, alternatively



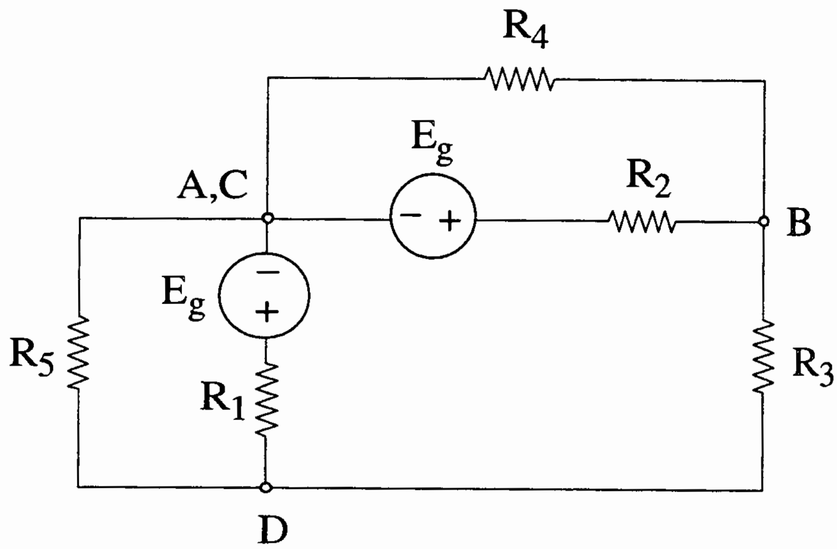
EXAMPLE 1



After a source transformation



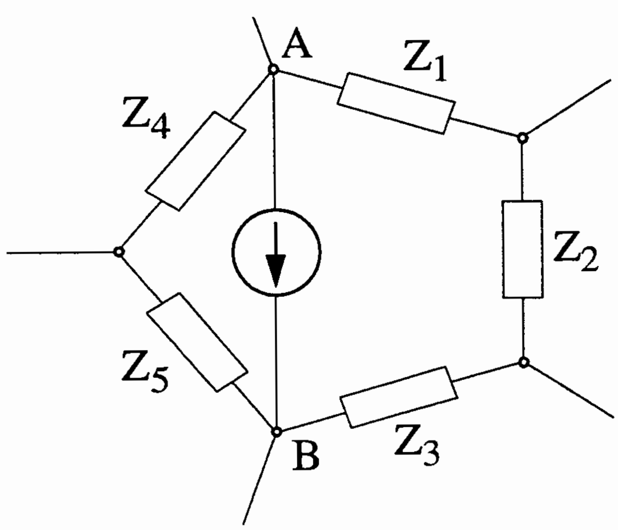
Since nodes A and C are shorted the circuit can be redrawn as



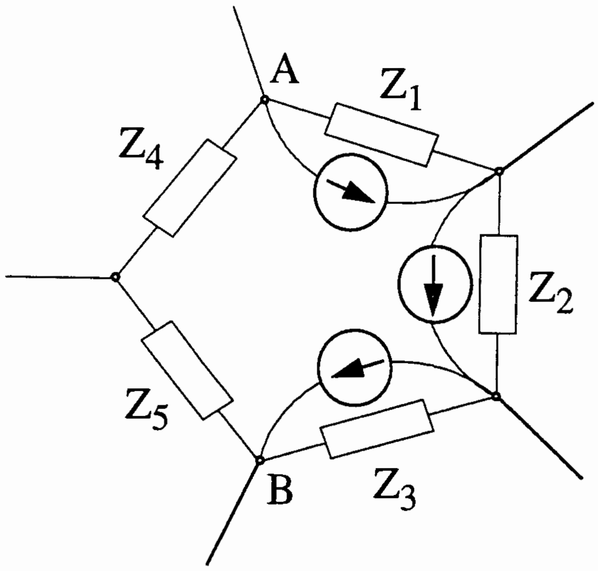
CURRENT SOURCE TRANSFORMATIONS

These transformations are necessary when there are no elements in parallel with an independent current source.

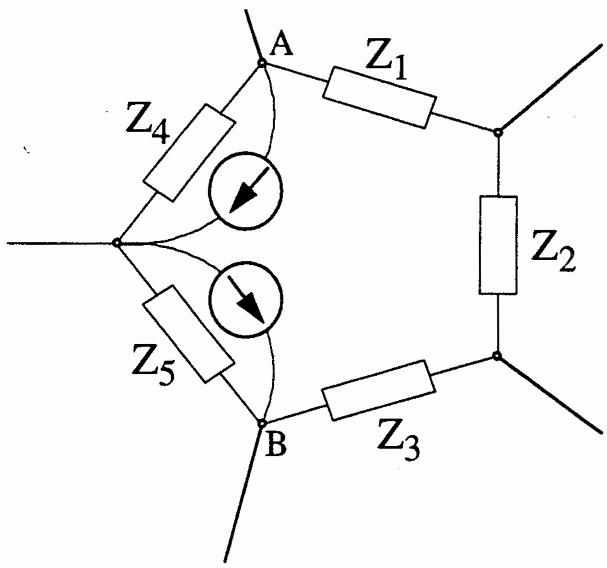
Generic situation



Equivalent representations

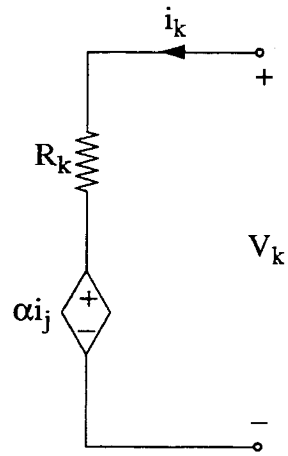


or, alternatively



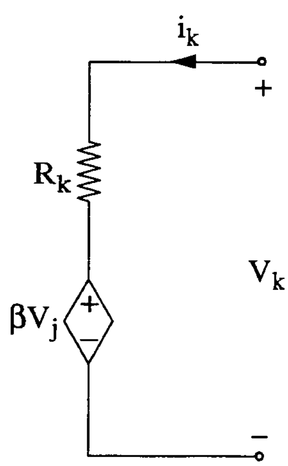
CONTROLLED SOURCES

Current controlled voltage source (CCVS)



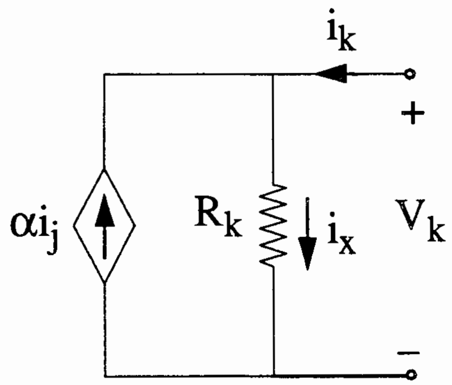
$$V_k(s) = R_k I_k(s) + \alpha I_j(s)$$

Voltage controlled voltage source (VCVS)



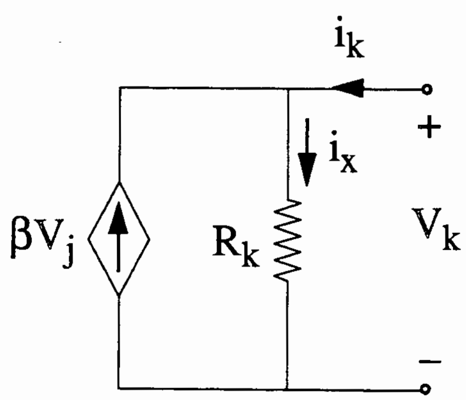
$$V_k(s) = R_k I_k(s) + \beta R_j I_j(s)$$

Current controlled current source (CCCS)



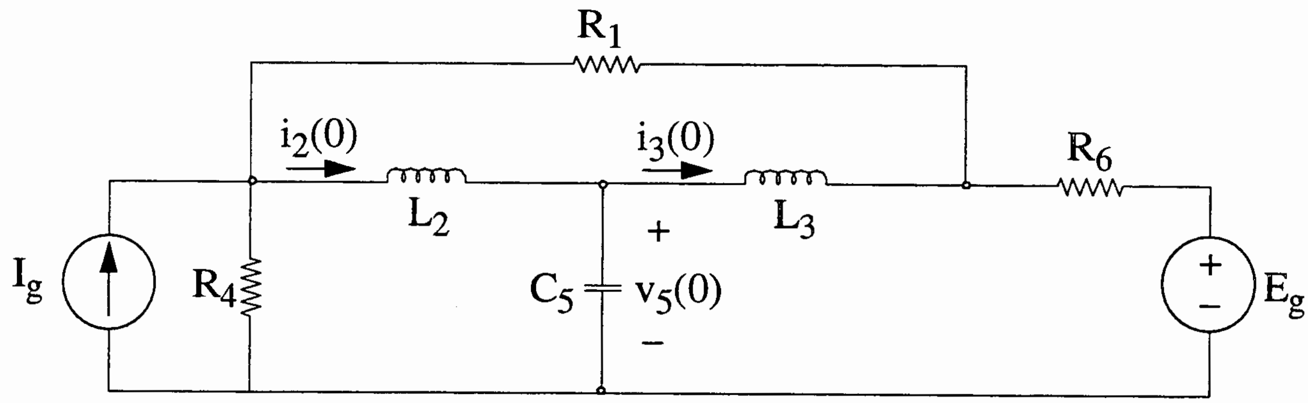
$$V_k(s) = R_k I_k(s) + \alpha R_k I_j(s)$$

Voltage controlled current source (VCCS)

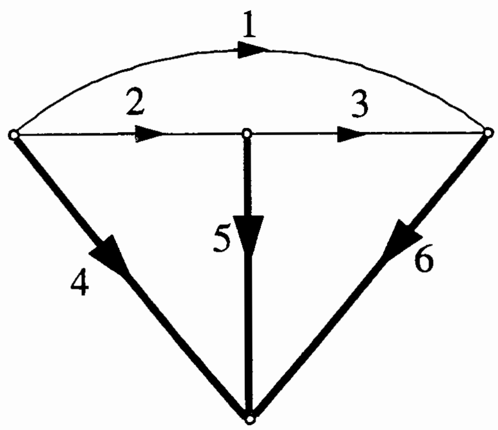


$$V_k(s) = R_k I_k(s) + \beta R_k R_j I_j(s)$$

EXAMPLE 2



Graph



Current-voltage relationships

(1) $V_1(s) = R_1 I_1(s)$

(2) $V_2(s) = s L_2 I_2(s) - L_2 i_2(0)$

(3) $V_3(s) = s L_3 I_3(s) - L_3 i_3(0)$

$$(4) \qquad V_4(s) = R_4 I_4(s) + R_4 I_g(s)$$

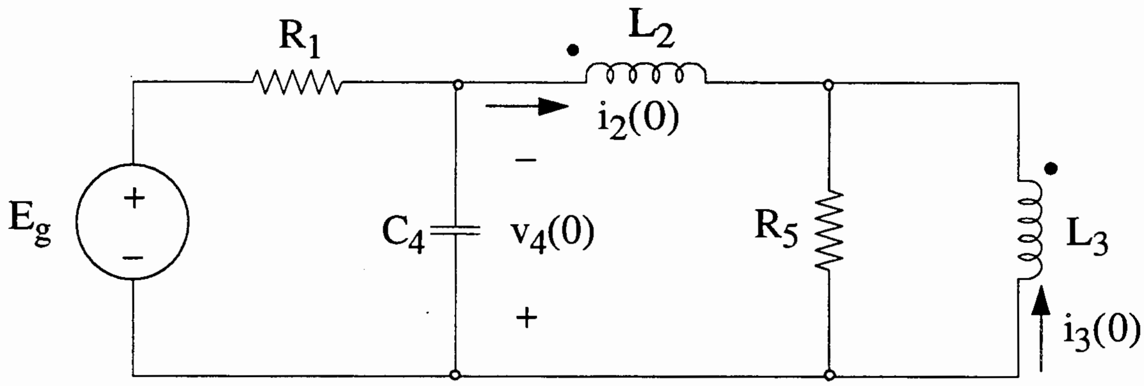
$$(5) \qquad V_5(s) = \frac{1}{sC_5} I_5(s) + \frac{v_5(0)}{s}$$

$$(6) \qquad V_6(s) = R_6 I_6(s) + E_g(s)$$

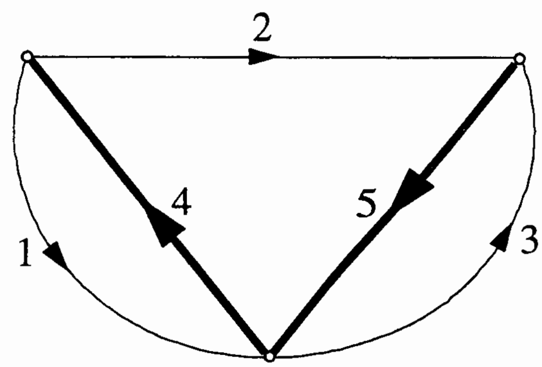
$$\begin{bmatrix} V_1(s) \\ V_2(s) \\ V_3(s) \\ V_4(s) \\ V_5(s) \\ V_6(s) \end{bmatrix} = \begin{bmatrix} R_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & sL_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & sL_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & R_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{sC_5} & 0 \\ 0 & 0 & 0 & 0 & 0 & R_6 \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \\ I_3(s) \\ I_4(s) \\ I_5(s) \\ I_6(s) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ R_4 I_g(s) \\ 0 \\ E_g(s) \end{bmatrix} -$$

$$- \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & L_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_1(0) \\ i_2(0) \\ i_3(0) \\ i_4(0) \\ i_5(0) \\ i_6(0) \end{bmatrix} + \frac{1}{s} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ v_5(0) \\ 0 \end{bmatrix}$$

EXAMPLE 3



Graph



Current-voltage relationships

(1) $V_1(s) = R_1 I_1(s) + E_g(s)$

(2) $V_2(s) = sL_2 I_2(s) - L_2 i_2(0) - sMI_3(s) + Mi_3(0)$

(3) $V_3(s) = sL_3 I_3(s) - L_3 i_3(0) - sMI_2(s) + Mi_2(0)$

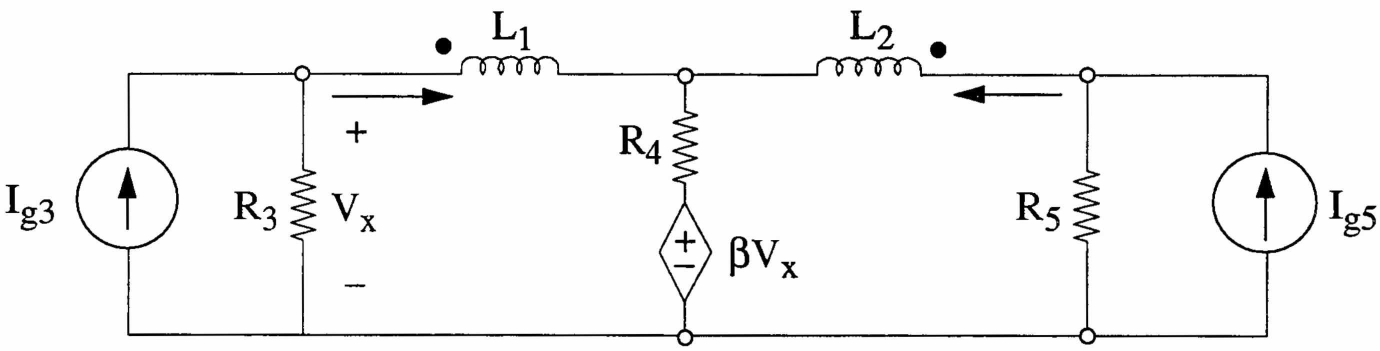
(4) $V_4(s) = \frac{1}{sC_4} I_4(s) + \frac{v_4(0)}{s}$

(5) $V_5(s) = R_5 I_5(s)$

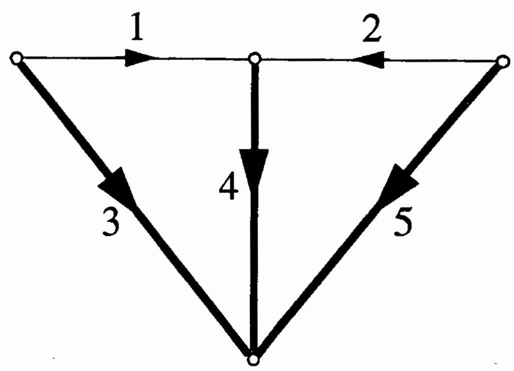
$$\begin{bmatrix} V_1(s) \\ V_2(s) \\ V_3(s) \\ V_4(s) \\ V_5(s) \end{bmatrix} = \begin{bmatrix} R_1 & 0 & 0 & 0 & 0 \\ 0 & sL_2 & -sM & 0 & 0 \\ 0 & -sM & sL_3 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{sC_4} & 0 \\ 0 & 0 & 0 & 0 & R_5 \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \\ I_3(s) \\ I_4(s) \\ I_5(s) \end{bmatrix} + \begin{bmatrix} E_g(s) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} -$$

$$- \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & L_2 & -M & 0 & 0 \\ 0 & -M & L_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_1(0) \\ i_2(0) \\ i_3(0) \\ i_4(0) \\ i_5(0) \end{bmatrix} + \frac{1}{s} \begin{bmatrix} 0 \\ 0 \\ 0 \\ v_4(0) \\ 0 \end{bmatrix}$$

EXAMPLE 4



Graph



Current-voltage relationships

(1) $V_1(s) = sL_1 I_1(s) - L_1 i_1(0) + sMI_2(s) - Mi_2(0)$

(2) $V_2(s) = sL_2 I_2(s) - L_2 i_2(0) + sMI_1(s) - Mi_1(0)$

(3) $V_3(s) = R_3 I_3(s) + R_3 I_{g3}(s)$

(4) $V_4(s) = R_4 I_4(s) + \beta V_3(s) = R_4 I_4(s) + \beta R_3 (I_3(s) + I_{g3}(s))$

(5) $V_5(s) = R_5 I_5(s) + R_5 I_{g5}(s)$

$$\begin{bmatrix} V_1(s) \\ V_2(s) \\ V_3(s) \\ V_4(s) \\ V_5(s) \end{bmatrix} = \begin{bmatrix} sL_1 & sM & 0 & 0 & 0 \\ sM & sL_2 & 0 & 0 & 0 \\ 0 & 0 & R_3 & 0 & 0 \\ 0 & 0 & \beta R_3 & R_4 & 0 \\ 0 & 0 & 0 & 0 & R_5 \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \\ I_3(s) \\ I_4(s) \\ I_5(s) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ R_3 I_{g3}(s) \\ \beta R_3 I_{g3}(s) \\ R_5 I_{g5}(s) \end{bmatrix} -$$

$$- \begin{bmatrix} L_1 & M & 0 & 0 & 0 \\ M & L_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_1(0) \\ i_2(0) \\ i_3(0) \\ i_4(0) \\ i_5(0) \end{bmatrix} + \frac{1}{s} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

LOOP AND CUTSET EQUATIONS

Basic ingredients

a) Kirchoff's laws:

$$B_f V_e(s) = 0 \qquad (KVL) \qquad V_e(s) \equiv \begin{bmatrix} V_l(s) \\ ---- \\ V_t(s) \end{bmatrix}$$

$$Q_f I_e(s) = 0 \qquad (KCL) \qquad I_e(s) \equiv \begin{bmatrix} I_l(s) \\ ---- \\ I_t(s) \end{bmatrix}$$

b) Current-voltage relationships:

$$V_e(s) = Z_e(s) I_e(s) + E_e(s) - L_e i_e(0) + \frac{1}{s} v_e(0)$$

and, if $Z_e^{-1}(s)$ exists,

$$I_e(s) = Z_e^{-1}(s) V_e(s) - Z_e^{-1}(s) \left\{ E_e(s) - L_e i_e(0) + \frac{1}{s} v_e(0) \right\}$$

c) Additional relationships:

$$I_e(s) = B_f^{\,T} I_l(s)$$

and

$$V_e(s) = Q_f^{\,T} V_t(s)$$

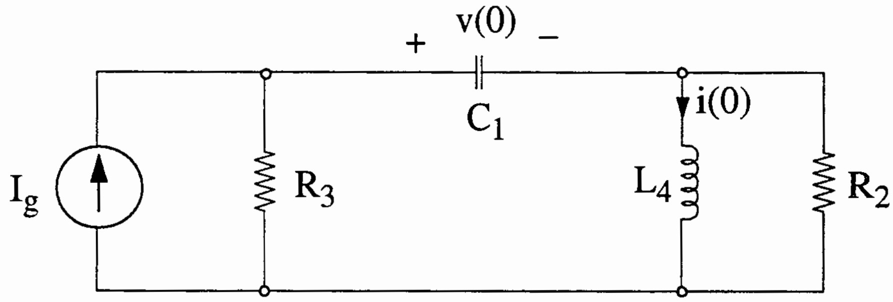
Loop Equations

$$\left[B_f\, Z_e(s)\, B_f^T\right] I_l(s) = -B_f\left\{E_e(s) - L_e i_e(0) + \frac{1}{s}\, v_e(0)\right\}$$

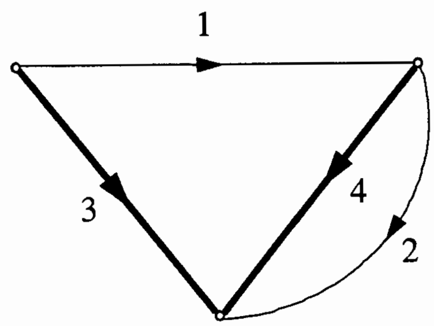
Cutset Equations

$$\left[Q_f\, Z_e^{-1}(s)\, Q_f^T\right] V_t(s) = Q_f\, Z_e^{-1}(s)\left\{E_e(s) - L_e i_e(0) + \frac{1}{s}\, v_e(0)\right\}$$

EXAMPLE 1



Graph



Network matrices

a) Cutset matrix

$$Q_f = \begin{array}{c} \\ c_1 \\ c_2 \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

b) Loop matrix

$$B_f = \begin{array}{c} \\ l_1 \\ l_2 \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \end{array}$$

Current-voltage relationships (CVR)

(1) $$V_1(s) = \frac{1}{sC_1} I_1(s) + \frac{1}{s} v_1(0)$$

(2) $$V_2(s) = R_2 I_2(s)$$

(3) $$V_3(s) = R_3 I_3(s) + R_3 I_g(s)$$

(4) $$V_4(s) = s L_4 I_4(s) - L_4 i_4(0)$$

In matrix form, CVR can be expressed as

$$\begin{bmatrix} V_1(s) \\ V_2(s) \\ V_3(s) \\ V_4(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{sC_1} & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 \\ 0 & 0 & R_3 & 0 \\ 0 & 0 & 0 & sL_4 \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \\ I_3(s) \\ I_4(s) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ R_3 I_g(s) \\ 0 \end{bmatrix} -$$

$$- \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_4 \end{bmatrix} \begin{bmatrix} i_1(0) \\ i_2(0) \\ i_3(0) \\ i_4(0) \end{bmatrix} + \frac{1}{s} \begin{bmatrix} v_1(0) \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Loop Equations

Left hand side

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{sC_1} & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 \\ 0 & 0 & R_3 & 0 \\ 0 & 0 & 0 & sL_4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 1 & -1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{sC_1} & 0 \\ 0 & R_2 \\ -R_3 & 0 \\ sL_4 & -sL_4 \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{sC_1} + R_3 + sL_4\right) & -sL_4 \\ -sL_4 & \left(R_2 + sL_4\right) \end{bmatrix}$$

Right hand side

$$\begin{bmatrix} -1 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{v_1(0)}{s} \\ 0 \\ R_3 I_g(s) \\ -L_4 i_4(0) \end{bmatrix} = \begin{bmatrix} -\frac{v_1(0)}{s} + R_3 I_g(s) + L_4 i_4(0) \\ -L_4 i_4(0) \end{bmatrix} =$$

Loop equations in final form:

$$\begin{bmatrix} \left(\frac{1}{sC_1} + R_3 + sL_4\right) & -sL_4 \\ -sL_4 & \left(R_2 + sL_4\right) \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = -\begin{bmatrix} -\frac{v_1(0)}{s} + R_3 I_g(s) + L_4 i_4(0) \\ -L_4 i_4(0) \end{bmatrix}$$

Cutset Equations

Left hand side

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} sC_1 & 0 & 0 & 0 \\ 0 & \frac{1}{R_2} & 0 & 0 \\ 0 & 0 & \frac{1}{R_3} & 0 \\ 0 & 0 & 0 & \frac{1}{sL_4} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} sC_1 & -sC_1 \\ 0 & \frac{1}{R_2} \\ \frac{1}{R_3} & 0 \\ 0 & \frac{1}{sL_4} \end{bmatrix} = \begin{bmatrix} \left(sC_1 + \frac{1}{R_3}\right) & -sC_1 \\ -sC_1 & \left(sC_1 + \frac{1}{R_2} + \frac{1}{sL_4}\right) \end{bmatrix}$$

Right hand side:

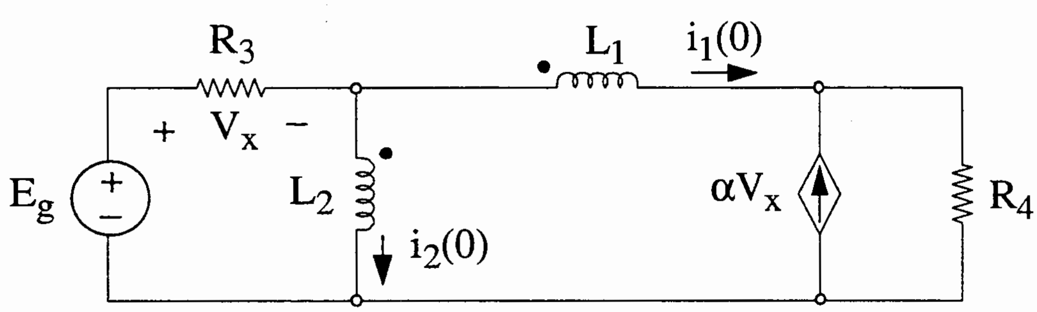
$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} sC_1 & 0 & 0 & 0 \\ 0 & \frac{1}{R_2} & 0 & 0 \\ 0 & 0 & \frac{1}{R_3} & 0 \\ 0 & 0 & 0 & \frac{1}{sL_4} \end{bmatrix} \begin{bmatrix} \frac{v_1(0)}{s} \\ 0 \\ R_3 I_g(s) \\ -L_4 i_4(0) \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 v_1(0) \\ 0 \\ I_g(s) \\ -\frac{1}{s} i_4(0) \end{bmatrix} = \begin{bmatrix} C_1 v_1(0) + I_g(s) \\ -C_1 v_1(0) - \frac{i_4(0)}{s} \end{bmatrix}$$

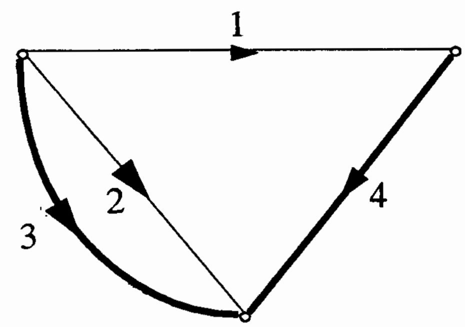
Cutset equations in final form:

$$\begin{bmatrix} \left(\frac{1}{R_3} + sC_1\right) & -sC_1 \\ -sC_1 & \left(sC_1 + \frac{1}{R_2} + \frac{1}{sL_4}\right) \end{bmatrix} \begin{bmatrix} V_3(s) \\ V_4(s) \end{bmatrix} = \begin{bmatrix} I_g(s) + C_1 v_1(0) \\ -C_1 v_1(0) - \frac{i_4(0)}{s} \end{bmatrix}$$

EXAMPLE 2 (loop equations only)



Graph



Loop matrix

$$B_f = \begin{array}{c} \\ l_1 \\ l_2 \end{array} \begin{bmatrix} e_1 & e_2 & e_3 & e_4 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

Current-voltage relationships (CVR)

(1) $V_1(s) = sL_1 I_1(s) - L_1 i_1(0) + sMI_2(s) - Mi_2(0)$

(2) $V_2(s) = sL_2 I_2(s) - L_2 i_2(0) + sMI_1(s) - Mi_1(0)$

(3) $V_3(s) = R_3 I_3(s) + E_g(s)$

(4) $V_4(s) = R_4 I_4(s) - \alpha R_3 R_4 I_3(s)$

In matrix form, CVR can be expressed as

$$\begin{bmatrix} V_1(s) \\ V_2(s) \\ V_3(s) \\ V_4(s) \end{bmatrix} = \begin{bmatrix} sL_1 & sM & 0 & 0 \\ sM & sL_2 & 0 & 0 \\ 0 & 0 & R_3 & 0 \\ 0 & 0 & -\alpha R_4 R_3 & R_4 \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \\ I_3(s) \\ I_4(s) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ E_g(s) \\ 0 \end{bmatrix} -$$

$$- \begin{bmatrix} L_1 & M & 0 & 0 \\ M & L_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_1(0) \\ i_2(0) \\ i_3(0) \\ i_4(0) \end{bmatrix}$$

Loop equations

Left hand side

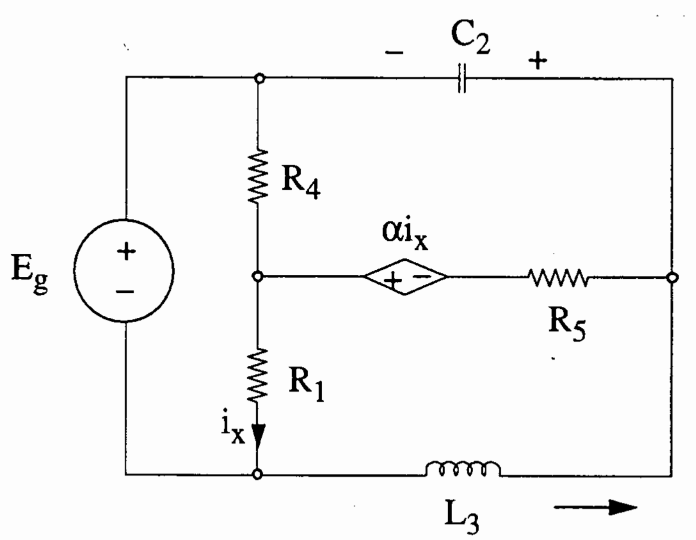
$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} sL_1 & sM & 0 & 0 \\ sM & sL_2 & 0 & 0 \\ 0 & 0 & R_3 & 0 \\ 0 & 0 & -\alpha R_4 R_3 & R_4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ 1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} (sL_1 + R_3 + \alpha R_4 R_3) & (sM + R_3 + \alpha R_4 R_3) \\ (sM + R_3) & (sL_2 + R_3) \end{bmatrix}$$

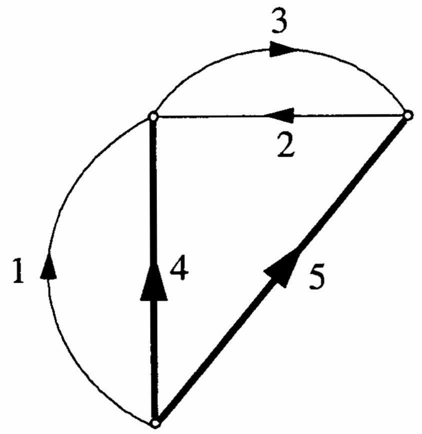
Right hand side

$$\begin{bmatrix} -1 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -L_1 i_1(0) - M i_2(0) \\ -M i_1(0) - L_2 i_2(0) \\ E_g(s) \\ 0 \end{bmatrix} = \begin{bmatrix} L_1 i_1(0) + M i_2(0) + E_g(s) \\ M i_1(0) + L_2 i_2(0) + E_g(s) \end{bmatrix}$$

EXAMPLE 3 (cutset equations only)



Graph (after source transformation)



Cutset Matrix

$$Q_f = \begin{matrix} \\ c_1 \\ c_2 \end{matrix} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 \\ \end{matrix}$$

$$Q_f = \begin{array}{c} c_1 \\ c_2 \end{array} \overset{\begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array}}{\begin{bmatrix} 1 & 1 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 \end{bmatrix}}$$

Current-voltage relationships (CVR)

(1) $V_1(s) = R_1 I_1(s)$

(2) $V_2(s) = E_g(s) + V_c(s) = E_g(s) + \dfrac{1}{sC_2} I_2(s) + \dfrac{v_c(0)}{s}$

$$(3) \qquad V_3(s) = sL_3 I_3(s) - L_3 i_3(0)$$

$$(4) \qquad V_4(s) = R_4 I_4(s) + E_g(s)$$

$$(5) \qquad V_5(s) = R_5 I_5(s) + \alpha I_1(s)$$

In matrix form

$$\begin{bmatrix} V_1(s) \\ V_2(s) \\ V_3(s) \\ V_4(s) \\ V_5(s) \end{bmatrix} = \begin{bmatrix} R_1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{sC_2} & 0 & 0 & 0 \\ 0 & 0 & sL_3 & 0 & 0 \\ 0 & 0 & 0 & R_4 & 0 \\ \alpha & 0 & 0 & 0 & R_5 \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \\ I_3(s) \\ I_4(s) \\ I_5(s) \end{bmatrix} + \begin{bmatrix} 0 \\ E_g(s) \\ 0 \\ E_g(s) \\ 0 \end{bmatrix} -$$

$$- \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & L_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_1(0) \\ i_2(0) \\ i_3(0) \\ i_4(0) \\ i_5(0) \end{bmatrix} + \frac{1}{s} \begin{bmatrix} 0 \\ v_c(0) \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The matrix $Z_e^{-1}(s)$ is

$$Z_e^{-1}(s) = \begin{bmatrix} \frac{1}{R_1} & 0 & 0 & 0 & 0 \\ 0 & sC_2 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{sL_3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{R_4} & 0 \\ -\frac{\alpha}{R_1 R_5} & 0 & 0 & 0 & \frac{1}{R_5} \end{bmatrix}$$

Cutset Equations

Left hand side

$$\begin{bmatrix} 1 & 1 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{R_1} & 0 & 0 & 0 & 0 \\ 0 & sC_2 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{sL_3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{R_4} & 0 \\ -\frac{\alpha}{R_1 R_5} & 0 & 0 & 0 & \frac{1}{R_5} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \left(\frac{1}{R_1} + sC_2 + \frac{1}{sL_3} + \frac{1}{R_4}\right) & -\left(sC_2 + \frac{1}{sL_3}\right) \\ -\left(sC_2 + \frac{1}{sL_3} + \frac{\alpha}{R_1 R_5}\right) & \left(sC_2 + \frac{1}{sL_3} + \frac{1}{R_5}\right) \end{bmatrix}$$

Right hand side

$$Q_f * Z_e^{-1}(s) * \begin{bmatrix} 0 \\ E_g(s) + \frac{1}{s} v_c(0) \\ -L_3 i_3(0) \\ E_g(s) \\ 0 \end{bmatrix} = Q_f * \begin{bmatrix} 0 \\ sC_2 E_g(s) + C_2 v_c(0) \\ -\frac{i_3(0)}{s} \\ \frac{E_g(s)}{R_4} \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} sC_2 E_g(s) + C_2 v_c(0) + \frac{i_3(0)}{s} + \frac{E_g(s)}{R_4} \\ -sC_2 E_g(s) - \frac{i_3(0)}{s} - C_2 v_c(0) \end{bmatrix}$$

STATE EQUATIONS

Advantages

1) Suitable for computation, since their solution does *not* require Laplace transforms.

2) They can be applied to *both* linear *and* nonlinear circuits.

Format for linear circuits

$$\dot{x} = Ax + Bu$$

Format for non-linear circuits

$$\dot{x} = f(x, u)$$

NON-DEGENERATE CIRCUITS

Non-degenerate circuits are circuits that have *no* E_g - C loops and *no* I_g - L nodes. In this case, the state variables are *all* inductor currents and *all* capacitor voltages.

Rules for formulating state equations

1) Place all voltage sources and capacitors *into* the tree.

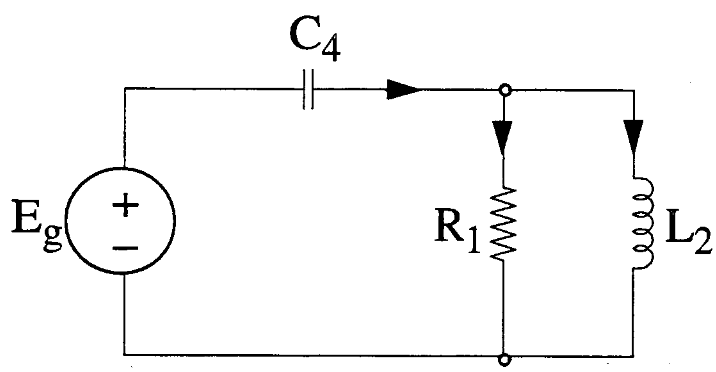
2) Place all current source and inductors *out* of the tree.

3) Write KCL for all fundamental cutsets involving capacitors.

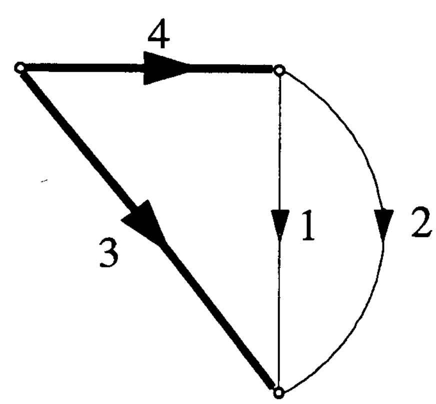
4) Write KVL for all fundamental loops involving inductors.

5) If any non-state variables appear on the right hand side, write additional KCL and KVL equations to eliminate them.

EXAMPLE 1



Graph



The basic KCL and KVL equations are

$$-i_{C_4} + i_{R_1} + i_{L_2} = 0$$

$$v_{L_2} - E_g + v_{C_4} = 0$$

In differential form :

$$C_4 \frac{dv_{C_4}}{dt} = i_{R_1} + i_{L_2}$$

$$L_2 \frac{di_{L_2}}{dt} = -v_{C_4} + E_g$$

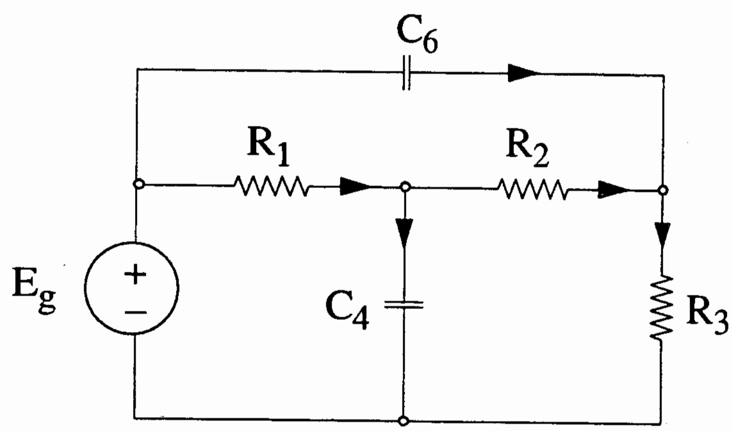
To eliminate i_{R1}, we need an additional KVL equation

$$v_{R_1} - E_g + v_{C_4} = 0 \quad \Rightarrow \quad i_{R_1} = \frac{E_g - v_{C_4}}{R_1}$$

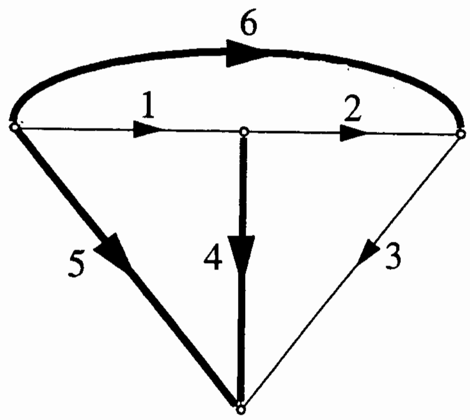
The state equations in matrix form are

$$\begin{bmatrix} \dfrac{dv_{C_4}}{dt} \\ \dfrac{di_{L_2}}{dt} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{C_4R_1} & \dfrac{1}{C_4} \\ -\dfrac{1}{L_2} & 0 \end{bmatrix} \begin{bmatrix} v_{C_4} \\ i_{L_2} \end{bmatrix} + \begin{bmatrix} \dfrac{1}{C_4R_1} \\ \dfrac{1}{L_2} \end{bmatrix} E_g$$

EXAMPLE 2



Graph



The basic KCL and KVL equations are

$$-i_{R_1} + i_{C_4} + i_{R_2} = 0$$

$$-i_{C_6} - i_{R_2} + i_{R_3} = 0$$

In differential form :

$$C_4 \frac{dv_{C_4}}{dt} = i_{R_1} - i_{R_2}$$

$$C_6 \frac{dv_{C_6}}{dt} = i_{R_3} - i_{R_2}$$

To eliminate i_{R1}, i_{R2} and i_{R3} we need three additional KVL equations

$$v_{R_1} + v_{C_4} - E_g = 0 \quad \Rightarrow \quad i_{R_1} = \frac{E_g - v_{C_4}}{R_1}$$

$$v_{R_2} - v_{C_6} + E_g - v_{C_4} = 0 \quad \Rightarrow \quad i_{R_2} = \frac{v_{C_6} + v_{C_4} - E_g}{R_2}$$

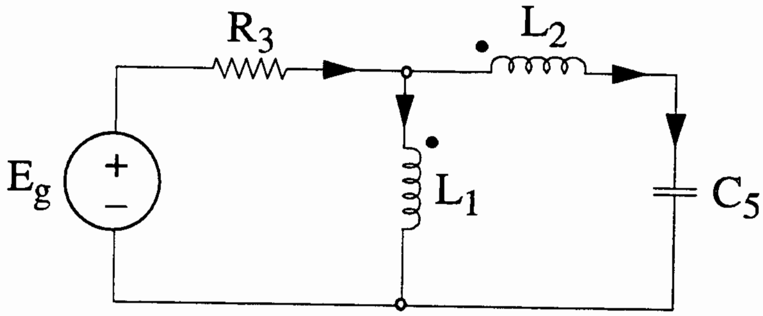
$$v_{R_3} - E_g + v_{C_6} = 0 \quad \Rightarrow \quad i_{R_3} = \frac{E_g - v_{C_6}}{R_3}$$

The state equations in matrix form are

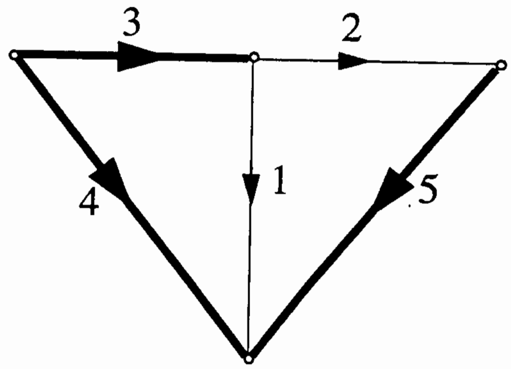
$$\begin{bmatrix} \dfrac{dv_{C_4}}{dt} \\ \dfrac{dv_{C_6}}{dt} \end{bmatrix} = \begin{bmatrix} -\left(\dfrac{1}{C_4R_1} + \dfrac{1}{C_4R_2}\right) & -\dfrac{1}{C_4R_2} \\ -\dfrac{1}{C_6R_2} & -\left(\dfrac{1}{C_6R_3} + \dfrac{1}{C_6R_2}\right) \end{bmatrix} \begin{bmatrix} v_{C_4} \\ v_{C_6} \end{bmatrix} +$$

$$+ \begin{bmatrix} \left(\dfrac{1}{C_4R_1} + \dfrac{1}{C_4R_2}\right) \\ \left(\dfrac{1}{C_6R_3} + \dfrac{1}{C_6R_2}\right) \end{bmatrix} E_g$$

EXAMPLE 3



Graph



The basic KCL and KVL equations are

$$i_{C_5} = i_{L_2}$$

$$v_{L_2} + v_{C_5} - E_g + v_{R_3} = 0$$

$$v_{L_1} - E_g + v_{R_3} = 0$$

To eliminate v_{R3}, we need an additional KCL equation

$$-i_{R_3} + i_{L_1} + i_{L_2} = 0 \quad \Rightarrow \quad v_{R_3} = R_3\,(i_{L_1} + i_{L_2})$$

Observing that there are coupled inductors in this circuit, we also have

$$v_{L_1} = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

$$v_{L_2} = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

Consequently, the state equations are

$$\begin{bmatrix} C_5 & 0 & 0 \\ 0 & L_1 & M \\ 0 & M & L_2 \end{bmatrix} \begin{bmatrix} \dfrac{dv_{C_5}}{dt} \\ \dfrac{di_{L_1}}{dt} \\ \dfrac{di_{L_2}}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -R_3 & -R_3 \\ -1 & -R_3 & -R_3 \end{bmatrix} \begin{bmatrix} v_{C_5} \\ i_{L_1} \\ i_{L_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} E_g$$

Note that in this case the state equations have the form

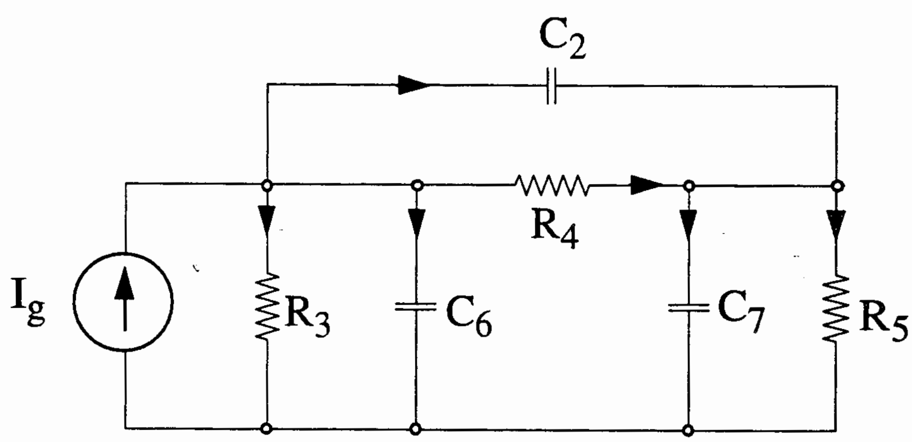
$$P\dot{x} = Ax + Bu$$

which can be reduced to the previous form as

$$\dot{x} = P^{-1}Ax + P^{-1}Bu$$

DEGENERATE CIRCUITS

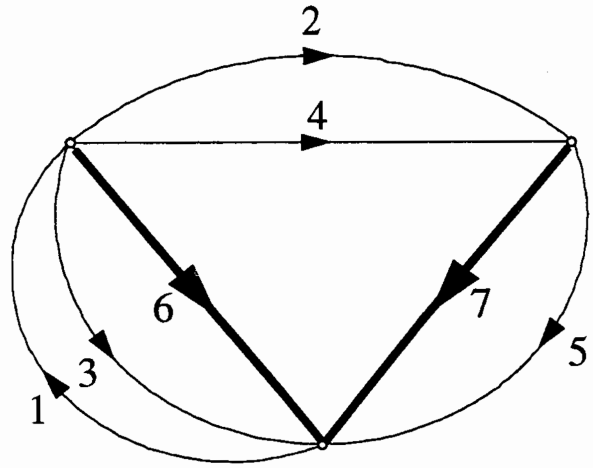
EXAMPLE 4



This circuit has a loop containing only capacitors C_2, C_6 and C_7, and is therefore degenerate. As a result, one of the capacitor voltages is *not independent* and should not be a state variable. In this example, we will treat C_2 as if it were a resistor (that is, we will not require that it belongs to the tree). To eliminate v_{c2} we will use the relationship

$$v_{C_2} = v_{C_6} - v_{C_7}$$

Graph



The basic KCL and KVL equations are

$$-i_g + i_{R_3} + i_{C_6} + i_{R_4} + i_{C_2} = 0$$

$$-i_{C_2} - i_{R_4} + i_{C_7} + i_{R_5} = 0$$

To eliminate i_{R3}, i_{R4} and i_{R5}, we use KVL equations

$$(1) \quad v_{R_3} = v_{C_6}$$

$$(2) \quad v_{R_4} = v_{C_6} - v_{C_7}$$

$$(3) \quad v_{R_5} = v_{C_7}$$

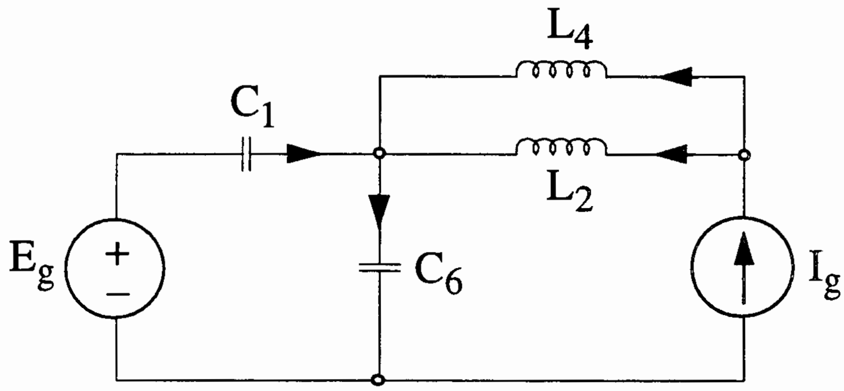
Note also that i_{C2} can be eliminated as

$$i_{C_2} = C_2 \frac{dv_{C_2}}{dt} = C_2 \frac{d}{dt}(v_{C_6} - v_{C_7})$$

The state equations in matrix form are

$$\begin{bmatrix} (C_2 + C_6) & -C_2 \\ -C_2 & (C_2 + C_7) \end{bmatrix} \begin{bmatrix} \frac{dv_{C_6}}{dt} \\ \frac{dv_{C_7}}{dt} \end{bmatrix} = \begin{bmatrix} -\left(\frac{1}{R_4} + \frac{1}{R_3}\right) & \frac{1}{R_4} \\ \frac{1}{R_4} & -\left(\frac{1}{R_4} + \frac{1}{R_5}\right) \end{bmatrix} \begin{bmatrix} v_{C_6} \\ v_{C_7} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i_g$$

EXAMPLE 5



In this circuit there is one loop involving only E_g, C_1 and C_6, and one node invloving only I_g, L_2 and L_4. We have degeneracy of *degree two*, so our state variables will be only v_{C6} and i_{L2}.

The degenerate loops and cutsets give us

$$v_{C_1} = E_g - v_{C_6}$$

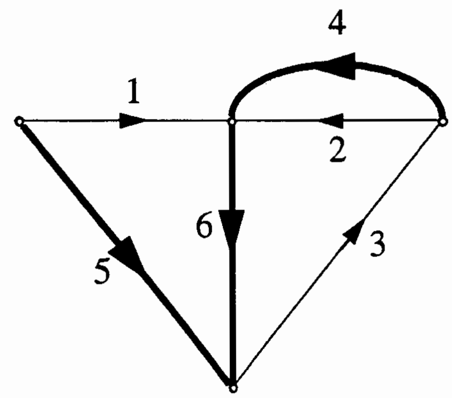
$$i_{L_4} = i_g - i_{L_2}$$

Therefore, we can eliminate i_{C1} and v_{L4} as

$$i_{C_1} = C_1 \frac{d}{dt}(E_g - v_{C_6})$$

$$v_{L_4} = L_4 \frac{d}{dt}(i_g - i_{L_2})$$

Graph



The basic KCL and KVL equations are

$$i_{C_6} - i_{C_1} - i_g = 0$$

$$v_{L_2} = v_{L_4}$$

The resulting state equations will be

$$\left(C_6 + C_1\right)\frac{dv_{C_6}}{dt} = i_g + C_1\frac{dE_g}{dt}$$

$$\left(L_2 + L_4\right)\frac{di_{L_2}}{dt} = L_4\frac{di_g}{dt}$$

Note that here we have an even more general format for state equations

$$P\dot{x} = Ax + Bu + F\dot{u}$$

This type of situation can arise only in degenerate circuits.