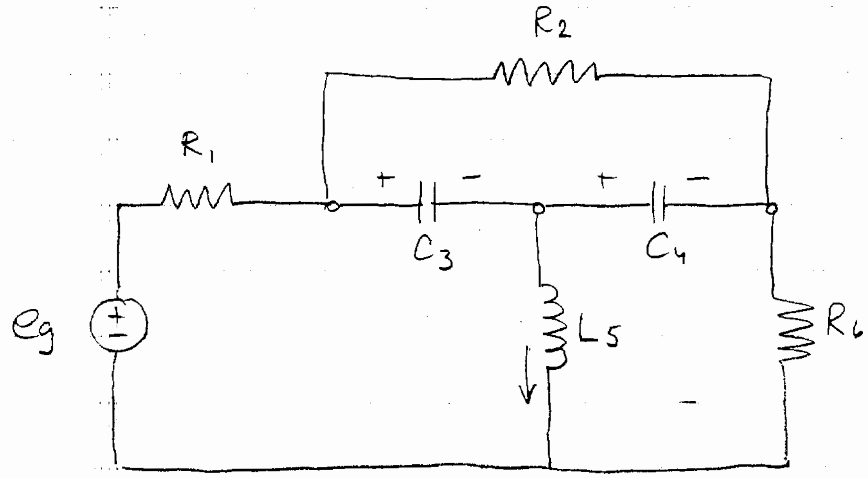
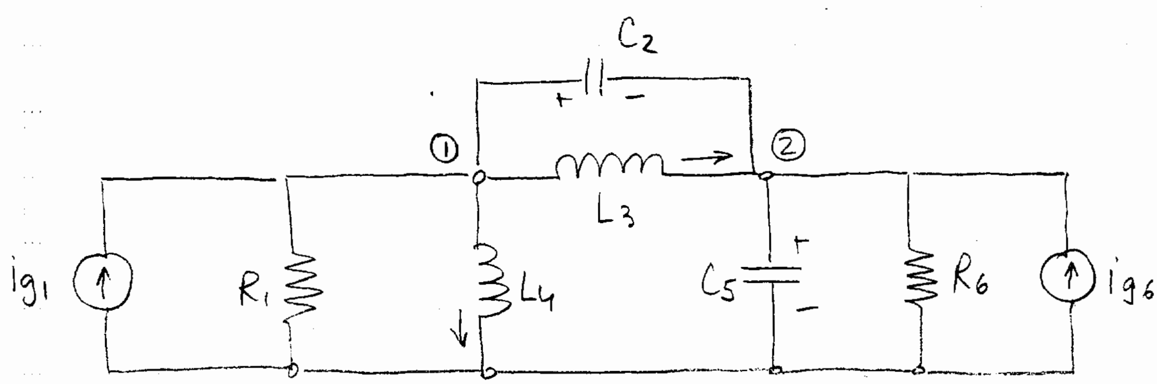
# HOMEWORK #4

① For the circuit shown below




pick the tree that contains the two capacitors and the inductor, and formulate the fundamental loop equations in matrix form

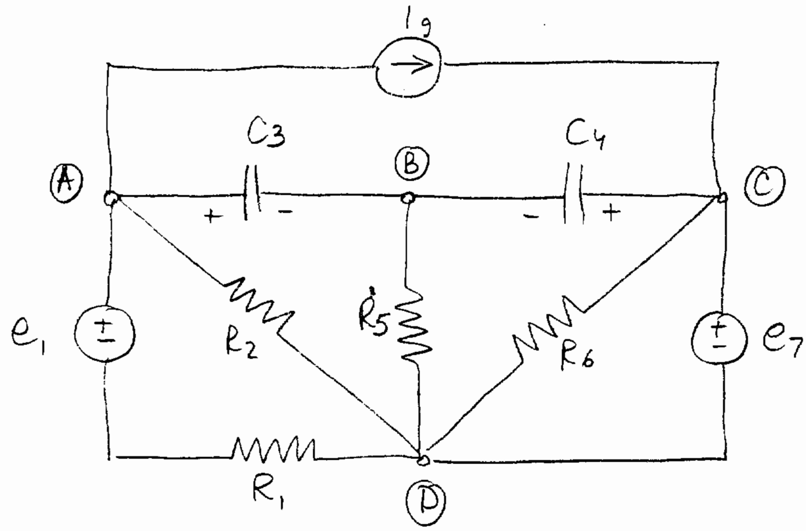
② For the circuit below




pick a tree that contains inductor $L_4$ and capacitor $C_5$. Then, formulate the fundamental cutset equations.

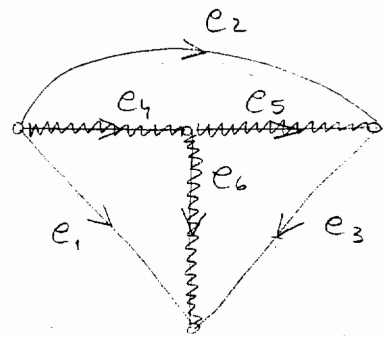
③ Repeat Problem 2 assuming that inductors $L_3$ and $L_4$ are coupled. The dot associated with $L_4$ is at node 1, and the dot for $L_3$ is at node 2.

④ Obtain the fundamental loop equations for the circuit shown below.



# SOLUTIONS TO HMWK #4

(1)




$$B_f = \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right] \end{array}$$

$$V_1(s) = R_1 I_1(s) + E_g(s)$$
$$V_2(s) = R_2 I_2(s)$$
$$V_3(s) = R_6 I_3(s)$$
$$V_4(s) = \frac{1}{sC_3} I_4(s) + \frac{V_{C3}(0)}{s}$$
$$V_5(s) = \frac{1}{sC_4} I_5(s) + \frac{V_{C4}(0)}{s}$$
$$V_6(s) = sL_5 I_6(s) - L_5 i_{L5}(0)$$

$$Z_e(s) = \begin{bmatrix} R_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & R_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{sC_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{sC_4} & 0 \\ 0 & 0 & 0 & 0 & 0 & sL_5 \end{bmatrix} \; ; \quad \begin{bmatrix} E_g(s) \\ 0 \\ 0 \\ \frac{V_{C3}(0)}{s} \\ \frac{V_{C4}(0)}{s} \\ -L_5 i_{L5}(0) \end{bmatrix}$$

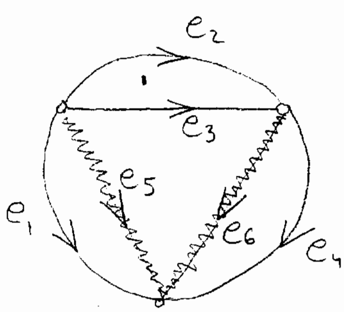
Forming

$$\left[B_f Z_e(s) B_f^T\right] I_\ell(s) = -B_f\left\{E_e(s) - L_e i_e(0) + \frac{1}{s}V_e(0)\right\}$$

we obtain

$$\begin{bmatrix} (R_1 + \frac{1}{sC_3} + sL_5) & \frac{1}{sC_3} & sL_5 \\ \frac{1}{sC_3} & (R_2 + \frac{1}{sC_3} + \frac{1}{sC_4}) & -\frac{1}{sC_4} \\ sL_5 & -\frac{1}{sC_4} & (R_6 + \frac{1}{sC_4} + sL_5) \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \\ I_3(s) \end{bmatrix} = \begin{bmatrix} -E_g + \frac{V_{c3}(0)}{s} - L_5 i_{L5}(0) \\ \frac{V_{c3}(0)}{s} + \frac{V_{c4}(0)}{s} \\ -\frac{V_{c4}(0)}{s} - L_5 i_{L5}(0) \end{bmatrix}$$

(2)




$$Q_f = \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \left[\begin{array}{cccc|cc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 1 \end{array}\right] \end{array}$$

$V_1(s) = R_1 I_1(s) + R_1 I_{g1}(s)$

$V_2(s) = \frac{1}{sC_2} I_2(s) + V_{c2}(0)/s$

$V_3(s) = sL_3 I_3(s) - L_3 i_{L3}(0)$

$V_4(s) = R_6 I_4(s) + R_6 I_{g6}(s)$

$V_5(s) = sL_4 I_5(s) - L_4 i_{L4}(0)$

$V_6(s) = \frac{1}{sC_5} I_6(s) + \frac{V_{c5}(0)}{s}$

$$Z_e(S) = \begin{bmatrix} R_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{SC_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & SL_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & R_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & SL_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{SC_5} \end{bmatrix} ; \quad \begin{bmatrix} R_1 I_{g1}(S) \\ V_{C2}(0)/S \\ -L_3 i_{L3}(0) \\ R_6 I_{g6}(S) \\ -L_4 i_{L4}(0) \\ V_{C5}(0)/S \end{bmatrix}$$

The inverse of $Z_e(S)$ is

$$Z_e^{-1}(S) = \begin{bmatrix} \frac{1}{R_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & SC_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{SL_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{R_6} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{SL_4} & 0 \\ 0 & 0 & 0 & 0 & 0 & SC_5 \end{bmatrix}$$

Forming

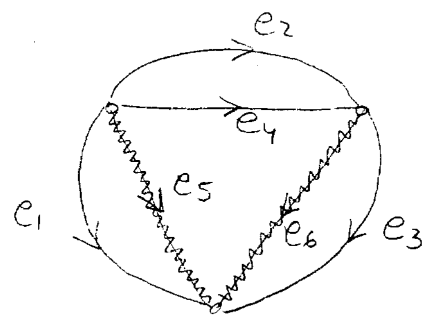
$$\left[ Q_f Z_e^{-1}(S) Q_f^T \right] V_T(S) = Q_f Z_e^{-1}(S) \left\{ E_e(S) - L_e i_e(0) + \frac{1}{S} V_e(0) \right\}$$

we have:

$$\begin{bmatrix} \left(\frac{1}{R_1} + SC_2 + \frac{1}{SL_3} + \frac{1}{SL_4}\right) & -\left(SC_2 + \frac{1}{SL_3}\right) \\ -\left(SC_2 + \frac{1}{SL_3}\right) & \left(SC_2 + \frac{1}{SL_3} + \frac{1}{R_6} + SC_5\right) \end{bmatrix} \begin{bmatrix} V_5(S) \\ V_6(S) \end{bmatrix} =$$

$$= \begin{bmatrix} I_{g1}(S) + C_2 V_{C2}(0) - \frac{i_{L3}(0)}{S} - \frac{i_{L4}(0)}{S} \\ -C_2 V_{C2}(0) + \frac{i_{L3}(0)}{S} + I_{g6}(S) + C_5 V_{C5}(0) \end{bmatrix}$$

3. In this case, it is convenient to number the two inductive branches <u>consecutively</u> (the matrix inversion is easier).




$$Q_f = \begin{array}{c} \\ \end{array}\begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array}$$
$$Q_f = \left[\begin{array}{cccc|cc} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$V_1(s) = R_1 I_1(s) + R_1 I_{g1}(s)$$
$$V_2(s) = \frac{1}{sC_2} I_2(s) + V_{C2}(0)/s$$
$$V_3(s) = R_6 I_3(s) + R_6 I_{g6}(s)$$
$$V_4(s) = sL_3 I_4(s) - L_3 i_{L3}(0) - sM I_5(s) + M i_{L5}(0)$$
$$V_5(s) = sL_4 I_5(s) - L_4 i_{L5}(0) - sM I_4(s) + M i_{L3}(0)$$
$$V_6(s) = \frac{1}{sC_5} I_6(s) + V_{C5}(0)/s$$

$$Z_e(s) = \begin{bmatrix} R_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{sC_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & R_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & sL_3 & -sM & 0 \\ 0 & 0 & 0 & -sM & sL_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{sC_5} \end{bmatrix} ; \quad \begin{bmatrix} R_1 I_{g1}(s) \\ V_{C2}(0)/s \\ R_6 I_{g6}(s) \\ -L_3 i_{L3}(0) + M i_{L5}(0) \\ -L_4 i_{L5}(0) + M i_{L3}(0) \\ V_{C5}(0)/s \end{bmatrix}$$

Before inverting $Z_e(s)$, we should take advantage of the fact that there is a 2×2 diagonal block in it (this is why consecutive numbering of inductive branches was convenient).

$$\begin{bmatrix} sL_3 & -sM \\ -sM & sL_4 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{L_4}{\Gamma} & \frac{M}{\Gamma} \\ \frac{M}{\Gamma} & \frac{L_3}{\Gamma} \end{bmatrix}$$

where $\Gamma \equiv s(L_3L_4 - M^2)$. Now,

$$Z_e^{-1}(s) = \begin{bmatrix} \frac{1}{R_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & sC_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{R_6} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{L_4}{\Gamma} & \frac{M}{\Gamma} & 0 \\ 0 & 0 & 0 & \frac{M}{\Gamma} & \frac{L_3}{\Gamma} & 0 \\ 0 & 0 & 0 & 0 & 0 & sC_5 \end{bmatrix}$$

In this case

$$Q_f Z_e^{-1}(s) = \begin{bmatrix} \frac{1}{R_1} & sC_2 & 0 & \frac{(L_4+M)}{\Gamma} & \frac{(L_3+M)}{\Gamma} & 0 \\ 0 & -sC_2 & \frac{1}{R_6} & -\frac{L_4}{\Gamma} & -\frac{M}{\Gamma} & sC_5 \end{bmatrix}$$

Consequently,

$$\begin{bmatrix} \left(\frac{1}{R_1} + sC_2 + \frac{L_3+L_4+2M}{\Gamma}\right) & -\left(sC_2 + \frac{L_4+M}{\Gamma}\right) \\ -\left(sC_2 + \frac{L_4+M}{\Gamma}\right) & \left(sC_2 + \frac{1}{R_6} + sC_5 + \frac{L_4}{\Gamma}\right) \end{bmatrix} \equiv Q_f Z_e^{-1}(s) Q_f^T$$

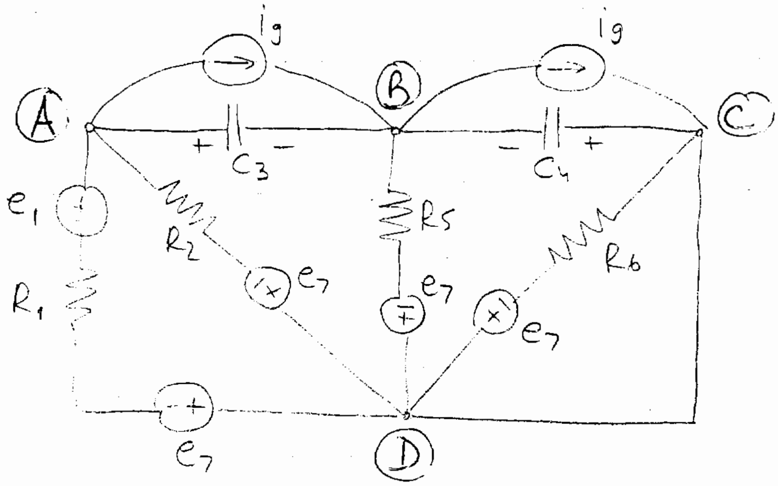
The right hand side gets rather messy:

$$RHS \equiv \begin{bmatrix} I_{g1}(s) + C_2 V_{c2}(0) + \left(\frac{L_4+M}{\Gamma}\right)\left(-L_3 i_{L3}(0) + M i_{L5}(0)\right) + \left(\frac{L_3+M}{\Gamma}\right)\left(-L_4 i_{L5}(0) + M i_{L3}(0)\right) \\ -C_2 V_{c2}(0) + I_{g6}(s) - \frac{L_4}{\Gamma}\left(-L_3 i_{L3}(0) + M i_{L5}(0)\right) - \frac{M}{\Gamma}\left(-L_4 i_{L5}(0) + M i_{L3}(0)\right) + C_5 V_{c5}(0) \end{bmatrix}$$

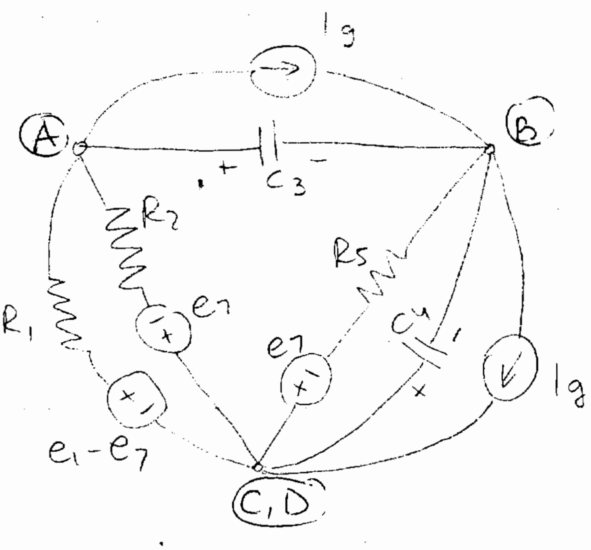
Note, however, that grouping the terms which multiply $i_{L3}(0)$ and $i_{L5}(0)$, and utilizing the fact that $\Gamma \equiv s(L_3L_4 - M^2)$, the RHS simplifies into:

$$RHS = \begin{bmatrix} I_{g1}(s) + C_2 V_{c2}(0) - \frac{i_{L3}(0)}{s} - \frac{i_{L5}(0)}{s} \\ -C_2 V_{c2}(0) + I_{g6}(s) + \frac{i_{L3}(0)}{s} + C_5 V_{c5}(0) \end{bmatrix}$$

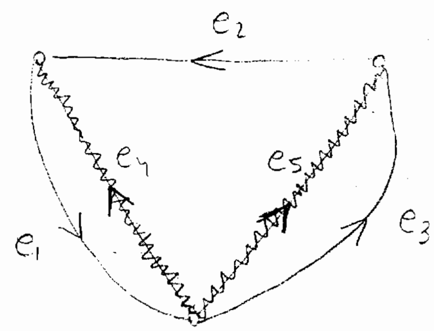
4 In this problem it is necessary to perform a transformation of both $e_7(t)$ and $i(t)$. The resulting circuit becomes:




Since nodes C and D are now shorted, the circuit can be redrawn as:




The corresponding graph will be




Note the directions, which are dictated by the sources!

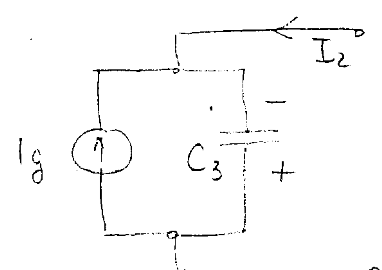
We have:

$$B_f = \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{array}\right] \end{array}$$

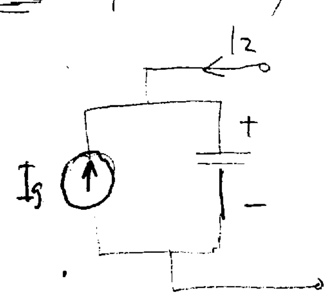
and

$$V_1(s) = R_1 I_1(s) + E_1(s) - E_7(s)$$
$$V_2(s) = \frac{1}{sC_3} I_2(s) + \frac{1}{sC_3} I_g(s) - \frac{V_{C3}(0)}{s}$$
$$V_3(s) = \frac{1}{sC_4} I_3(s) + \frac{1}{sC_4} I_g(s) + \frac{V_{C4}(0)}{s}$$
$$V_4(s) = R_2 I_4(s) + E_7(s)$$
$$V_5(s) = R_5 I_5(s) + E_7(s)$$

Note that in $V_2(s)$ we have $-\frac{V_{C3}(0)}{s}$. This is because in this branch:




and our standard expression was derived for the opposite polarity of $V_{C3}(0)$



Consequently,

$$
Ze(S) = \begin{bmatrix} R_1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{SC_3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{SC_4} & 0 & 0 \\ 0 & 0 & 0 & R_2 & 0 \\ 0 & 0 & 0 & 0 & R_5 \end{bmatrix} \; ; \quad \begin{bmatrix} E_1(S) - E_7(S) \\ \frac{1}{SC_3} I_g(S) - \frac{V_{C3}(0)}{S} \\ \frac{1}{SC_4} I_g(s) + \frac{V_{C4}(0)}{S} \\ E_7(S) \\ E_7(S) \end{bmatrix}
$$

The loop equations can now be written as:

$$
\begin{bmatrix} (R_1+R_2) & -R_2 & 0 \\ -R_2 & (\frac{1}{SC_3}+R_2+R_5) & -R_5 \\ 0 & -R_5 & (\frac{1}{SC_4}+R_5) \end{bmatrix} \begin{bmatrix} I_1(S) \\ I_2(S) \\ I_3(S) \end{bmatrix} = \begin{bmatrix} -E_1(S) \\ -\frac{1}{SC_3} I_g(S) + \frac{V_{C3}(0)}{S} \\ -\frac{1}{SC_4} I_g(S) - \frac{V_{C4}(0)}{S} + E_7(S) \end{bmatrix}
$$