ELEN 162
A. I. Zecevic

Project 2

1. Consider the graph shown below (which represents a simplified model of the New England power network).

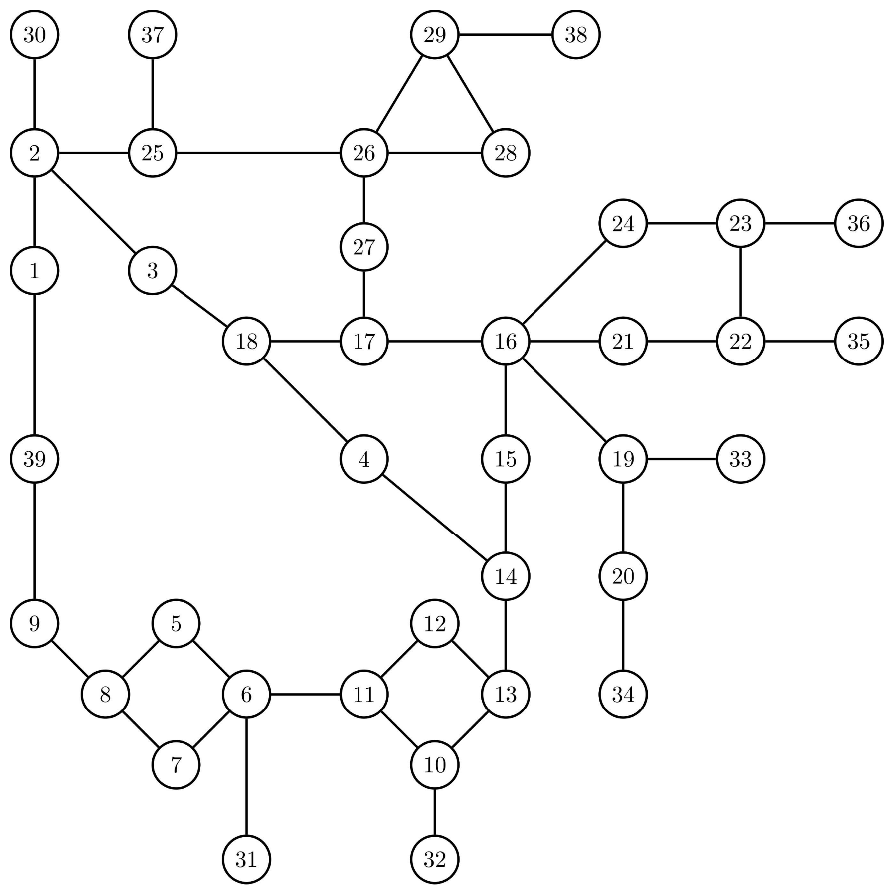

Figure 1: Model of the New England power network.

   (a) Form a symmetric matrix that corresponds to this graph. For the sake of simplicity, you can assume that all the diagonal elements are 10, and that all off diagonal elements equal 0.1.

   (b) Perform an LU factorization of this matrix, and record the number of fill in elements (you can calculate this number as $\text{nnz}(L + U) - \text{nnz}(A)$).

   (c) Obtain spy plots for matrices A and $L + U$. Do these matrices possess any kind of internal structure that could be exploited for parallelizing the factorization? Explain.

2. Determine a minimal degree ordering for the matrix obtained in Problem 1(a) *without* using Matlab. Apply the permutation vector that you obtained to reorder the matrix.

(a) Perform an LU factorization of the permuted matrix $\tilde{A}$, and plot matrix $L + U$. How does the number of fill in elements compare to what you found in Problem 1(b)?

(b) Use Matlab's function symamd(A) to obtain a symetric minimal degree ordering for matrix A. Use the corresponding permutation vector to reorder the matrix.

(c) Perform an LU factorization of the permuted matrix, and record the number of fill in elements. Does the Matlab ordering produce better results than the ordering you obtained "by hand" in part (a)? Explain.

3. Consider the matrix shown below, whose nonzero elements are indicated by $*$.

$$A = \begin{bmatrix} * & 0 & 0 & * & 0 & * & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & * & * & 0 & 0 & * & 0 & 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & 0 & 0 & * & 0 & 0 & 0 & * & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * & 0 & 0 & 0 & * \\ * & 0 & 0 & * & * & 0 & 0 & * & 0 & 0 & * & 0 & * & * & * \\ 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & * & 0 & * & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & * & 0 & 0 & 0 & * \\ 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & * & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & 0 & 0 & 0 & * \end{bmatrix} \qquad (1)$$

In the following, we will assume that you have 4 processors at your disposal, and your objective will be to form a nested BBD structure that allows us to efficiently parallelize the process of LU factorization.

(a) Draw the graph that corresponds to matrix A, and find its Laplacian Q.

(b) Compute the eigenvector that corresponds to the smallest positive eigenvalue of Q, and use it to find a minimal separator. Determine the correponding permutation vector, and show the permuted matrix (make sure that you explicitly identify the border and the diagonal blocks).

(c) Repeat the procedure that you performed in part (b) on each diagonal block. Indicate what the final permutation vector looks like, and show the resulting matrix.

4. In this problem we will consider the same matrix as in Problem 3, but you will now be asked to use nested dissection to obtain a BBD structure. As in the previous problem, you can assume that 4 processors are available.

(a) Find a pseudo-peripheral node for the graph that you obtained in Problem 3(a), using node 1 as you initial guess. **Note:** When choosing the root node for the next iteration, pick the node from the last level that has the *lowest* degree. In the case of a tie, use the node that has the smallest number.

(b) Determine the final rooted level structure, and use it to identify an appropriate separator. Indicate the correponding permutation vector, and show what the permuted matrix looks like.

(c) Repeat the procedure that you performed in part (b) on the diagonal blocks. Indicate what the final permutation vector looks like, and show the resulting matrix.

(d) Based on the results that you obtained in Problems 3 and 4, which of these two decompositions is better suited for parallel LU factorization using 4 processors? Explain your reasoning.

5. Consider a system of linear equations of the form

$$Ax = b \tag{2}$$

where

$$A = \begin{bmatrix} 6 & 0 & 4 & 0.3 & 0.4 & 0 & 0.1 & 0.1 \\ 0 & 8 & 0.6 & 0.2 & 0 & 0.2 & 0.3 & 6 \\ 4 & 0.6 & 5 & 0.3 & 0.1 & 0.2 & 0.1 & 0 \\ 0.3 & 0.2 & 0.3 & 6 & 0 & 2 & 0 & 0 \\ 0.4 & 0 & 0.1 & 0 & 4 & 0.2 & 2 & 0.3 \\ 0 & 0.2 & 0.2 & 2 & 0.2 & 4 & 0 & 0.2 \\ 0.1 & 0.3 & 0.1 & 0 & 2 & 0 & 6 & 0.1 \\ 0.1 & 6 & 0 & 0 & 0.3 & 0.2 & 0.1 & 8 \end{bmatrix}; \qquad b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \tag{3}$$

(a) Solve this system using the Jacobi iterative method, assuming that matrix Q has diagonal blocks of sizes 1×1, 2×2 and 4×4, respectively. For each of these three scenarios, record the number of iterations needed to satisfy the convergence condition

$$\|x(k) - F(x(k))\|_\infty < 10^{-4} \tag{4}$$

where $x(0) = 0$, and

$$F(x) = Gx + w \tag{5}$$

Note: Recall that G and w are defined as

$$G = I - Q^{-1}A \tag{6}$$

and

$$w = Q^{-1}b \tag{7}$$

for both the Jacobi and the Gauss-Sedel method.

(b) Repeat part (a) of this problem using the Gauss-Seidel method.

6. Perform an epsilon decomposition on matrix A from Problem 5 using an appropriately chosen value for ε. Show the bipartitite graphs that you obtained explicitly (as well as the permuted matrix).

(a) Repeat parts (a) and (b) of Problem 5 using the permuted matrix $\tilde{A}$ and the corresponding vector $\tilde{b}$.

(b) If you are asked to solve this problem using 4 processors, would you use the Jacobi method or the Gauss-Seidel method? Explain.