ELEN 160
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Project 1: The Logistic Map

The logistic map

$$x(k+1) = px(k)[1 - x(k)]$$

is often used as tool for studying chaotic dynamics. The applications of this difference equation range from finance and meteorology to fluid mechanics and ecology. A typical example is the modelling of moth colonies, in which case $x(k)$ represents the moth population in a single generation (scaled to a number between 0 and 1).

Problem 1. Write an m-file in Matlab that will compute the sequence $x(k)$, with $x(0)$, p and $N = k_{\max}$ as input parameters. Use this file to create a second m-file, which will produce the bifurcation diagram for the logistic map. Plot the bifurcation diagram for $0 \leq p \leq 4$, and identify four distinct regions that correspond to different types of dynamic behavior. Characterize this behavior explicitly.

Problem 2. For each of the four regions in Problem 1:

a) Plot a representative sequence $x(k)$ with $x(0) = 0.5$ as the starting point. Try this for $p > 4$ as well, and record what happens.

b) Repeat part *a*) with $x(0) = 1.5$, and observe what changes.

Problem 3.
a) In one of the four regions the system exhibits classic period doubling. Identify subregions that correspond to $T = 2, \ldots, 32$ and provide plots of $x(k)$ that illustrate this (use $x(0) = 0.5$ in all cases).

b) Arrange your results into the following table

Period	Region	Size of Region
$T = 2$	$p_1 \leq p \leq p_2$	$w_1 = p_2 - p_1$
$T = 4$	$p_3 \leq p \leq p_4$	$w_2 = p_4 - p_3$
$T = 8$	$p_5 \leq p \leq p_6$	$w_3 = p_6 - p_5$
$T = 16$	$p_7 \leq p \leq p_8$	$w_4 = p_8 - p_7$
$T = 32$	$p_9 \leq p \leq p_{10}$	$w_5 = p_{10} - p_9$

Based on this table, form ratios

$$F_1 = w_1/w_2$$
$$F_2 = w_2/w_3$$
$$F_3 = w_3/w_4$$
$$F_4 = w_4/w_5$$

If your work was sufficiently accurate, the values of $F_1, \ldots, F_4$ should be similar. Ideally (*i.e.* when $T \to \infty$) these ratios converge to a particular number which is known as the *Feigenbaum constant*. Use your results to estimate this number.

NOTE. The Feigenbaum constant is associated with a wide variety of mappings on the interval $[0, 1]$. This is quite unexpected, and gives this number a rather special status in mathematics (not unlike π or ϕ).

Problem 4. In a subset of the region $3.82 \leq p \leq 3.83$ the system exhibits intermittent dynamics. Find a value of p for which this behavior is obvious (by "obvious" I mean that there is only *one* brief disruption in the regular oscillations for $k = 0, 1, \ldots, 500$). Plot the corresponding $x(k)$ using $x(0) = 0.5$ as the initial condition. Note that this may require a very *precise* choice of p, involving as many as five decimal places.

Problem 5. Consider two chaotic solutions whose initial conditions differ by 10^{-8} (for the sake of uniformity, let one of them be $x(0) = 0.5$). Plot their *difference* as a function of k, and estimate how long it takes for the two solutions to become visibly distinct.