FUNDAMENTALS OF COMPUTER-AIDED CIRCUIT SIMULATION

LECTURE NOTES
FOR ELEN 118

Aleksandar I. Zecevic

Dept. of Electrical Engineering

Santa Clara University

SECTION I:

SPARSE MATRICES

SPARSE MATRICES

A *sparse matrix* is a matrix in which the great majority (typically 98% or more) of the elements are *zeros*. Such matrices arise in practically every engineering discipline.

Sparsity in circuits results from the fact that no matter how large a circuit is, any given node is connected to only a few other nodes (due to physical constraints).

Sparsity is a key feature of large scale circuits such as VLSI digital circuits or electric power networks.

Storage of sparse matrices

For sparse matrices, it is necessary to store *only* the non-zero entries. This results in enormous memory savings; for example, storing *all* the 2.5 million entries of a 5,000 × 5,000 matrix would require some 200Mb (assuming each entry, zero or non-zero, requires 8 bytes).

Storing a sparse matrix requires three vectors, typically denoted **B**, **JB** and **IB**.

1) Vector **B** stores all the non-zero *values* as a string.

2) Vector **JB** stores *column locations* of non-zero elements.

3) Vector **IB** stores a *pointer* to the start of each row.

EXAMPLE

Consider a 9×9 matrix A with twelve non-zero entries:

Diagonal entries : $A(1, 1) = A(2, 2) = \;.....\; = A(9, 9) = 1$

Off-diagonal entries: $A(1, 6) = 4;\; A(7, 3) = 2;\; A(9, 3) = 0.1$

Vector **B**

$$\begin{array}{rccccccccccccccccccc}
 & 1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & 8 & & 9 & \\
B = \Big[& 1\;\;4 & | & 1 & | & 1 & | & 1 & | & 1 & | & 1 & | & 2\;\;1 & | & 1 & | & 0.1\;\;1 & \Big]
\end{array}$$

Vector **JB**

$$\begin{array}{rccccccccccccccccccc}
 & 1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & 8 & & 9 & \\
JB = \Big[& 1\;\;6 & | & 2 & | & 3 & | & 4 & | & 5 & | & 6 & | & 3\;\;7 & | & 8 & | & 3\;\;9 & \Big]
\end{array}$$

Vector **IB**

$$\begin{array}{rccccccccccccccccccc}
 & 1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & 8 & & 9 & \\
IB = \Big[& 1 & | & 3 & | & 4 & | & 5 & | & 6 & | & 7 & | & 8 & | & 10 & | & 11 & \Big]
\end{array}$$

Recovering information for row 7 of A

Size of row 7:	**IB**(8) - **IB**(7) = 2 ⇒ row 7 has *two* nonzero elements.
Where to find these elements:	**IB**(7) = 8 ⇒ Starting loction is **B**(8). Since there are *two* nonzeros, they are in **B**(8) and **B**(9) respectively.
Column location:	**IB**(7) = 8 ⇒ Starting location is **JB**(8). Since there are *two* nonzeros, **JB**(8) and **JB**(9) contain column locations.

Summary of information for row 7

JB(8) = 3, JB(9) = 7 ⇒ A(7, 3) and A(7, 7) are nonzeros in row 7.

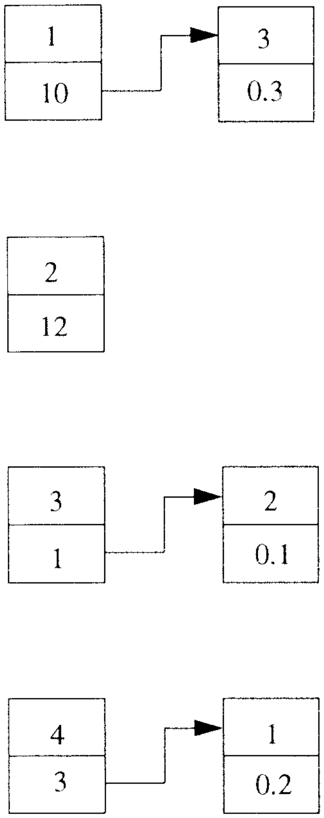
B(8) = 2; B(9) = 1 ⇒ A(7, 3) = 2 and A(7, 7) = 1.

An alternative storage technique

Data structures can also be used for efficient storage of sparse matrices.

EXAMPLE

$$A = \begin{bmatrix} 10 & 0 & 0.3 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0.1 & 1 & 0 \\ 0.2 & 0 & 0 & 3 \end{bmatrix}$$



Computation with sparse matrices

Basic computational problem in circuits - solving a large system of linear algebraic equations

$$A\,x = b$$

The most popular solution technique for circuit problems is based on *LU factorization.*

Solution procedure using LU factorization

1) Rewrite A as $A = LU$, where L is *lower triangular* and U is *upper triangular.*

2) Solve $L\,z = b$ for z.

3) Solve $U\,x = z$ for b.

EXAMPLE (Algorithm 1 for LU factorization)

$$A = \begin{bmatrix} 5 & 1 & 2 \\ 1 & 4 & 1 \\ 2 & 2 & 5 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} * \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

STEP 1

Compute the *first row* of U (using the fact that $l_{11} = 1$)

$$5 = a_{11} = l_{11}\,u_{11} \quad\Rightarrow\quad u_{11} = 5$$

$$1 = a_{12} = l_{11}\,u_{12} \quad\Rightarrow\quad u_{12} = 1$$

$$2 = a_{13} = l_{11}\,u_{13} \quad\Rightarrow\quad u_{13} = 2$$

Compute the *first column* of L (using the already computed u_{11})

$$1 = a_{21} = l_{21}\,u_{11} = 5l_{21} \quad\Rightarrow\quad l_{21} = 0.2$$

$$2 = a_{31} = l_{31}\,u_{11} = 5l_{31} \quad\Rightarrow\quad l_{31} = 0.4$$

Situation after Step 1

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0.2 & * & 0 \\ 0.4 & * & * \end{bmatrix} \quad ; \quad U = \begin{bmatrix} 5 & 1 & 2 \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$$

STEP 2

Compute the *second row* of U (using $l_{22} = 1$ as well as the computed values for u_{12} and u_{13})

$$4 = a_{22} = l_{21} u_{12} + l_{22} u_{22} \Rightarrow 3.8 = l_{22} u_{22} \Rightarrow u_{22} = 3.8$$

$$1 = a_{23} = l_{21} u_{13} + l_{22} u_{23} \Rightarrow 0.6 = l_{22} u_{23} \Rightarrow u_{23} = 0.6$$

Compute the *second column* of L (using the previously computed values for l_{31}, u_{12} and u_{22})

$$2 = a_{32} = l_{31} u_{12} + l_{32} u_{22} \Rightarrow 1.6 = 3.8 l_{32} \Rightarrow l_{32} = 0.421$$

Situation after Step 2

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0.2 & 1 & 0 \\ 0.4 & 0.421 & * \end{bmatrix} \quad ; \quad U = \begin{bmatrix} 5 & 1 & 2 \\ 0 & 3.8 & 0.6 \\ 0 & 0 & * \end{bmatrix}$$

STEP 3

Compute the *third row* of U (using the previously computed values for l_{31}, l_{32}, u_{13} and u_{23}, as well as $l_{33} = 1$)

$$5 = a_{33} = l_{31}\,u_{13} + l_{32}\,u_{23} + l_{33}\,u_{33} \Rightarrow 3.9474 = l_{32}\,u_{33} \Rightarrow u_{33} = 3.9474$$

FINAL SITUATION

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0.2 & 1 & 0 \\ 0.4 & 0.421 & 1 \end{bmatrix} \quad ; \quad U = \begin{bmatrix} 5 & 1 & 2 \\ 0 & 3.8 & 0.6 \\ 0 & 0 & 3.947 \end{bmatrix}$$

COMMENT: This algorithm for LU factorization uses previously computed elements of L and U *when they are needed*, not when they become available.

SAME EXAMPLE (Algorithm 2 for LU factorization)

In this algorithm we make use of previously computed elements of L and U *as soon as they become available.*

STEP 1

Exactly the same as in Algorithm 1, resulting in

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & * & 0 \\ l_{31} & * & * \end{bmatrix} \quad ; \quad \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$$

with $l_{11} = 1$; $l_{21} = 0.2$; $l_{31} = 0.4$; $u_{11} = 5$; $u_{12} = 1$; $u_{13} = 2$

STEP 2

a) Form 2×2 matrix W_2

$$W_2 = \begin{bmatrix} l_{21} \\ l_{31} \end{bmatrix} \begin{bmatrix} u_{12} & u_{13} \end{bmatrix} \equiv \begin{bmatrix} l_{21}\, u_{21} & l_{21}\, u_{13} \\ l_{31}\, u_{12} & l_{31}\, u_{13} \end{bmatrix}$$

Note that all computed elements of L and U are now used *as soon as they become available.*

b) Form 2 × 2 matrix A_2

$$A_2 \equiv \begin{bmatrix} a_{22}^{(2)} & a_{23}^{(2)} \\ a_{32}^{(2)} & a_{33}^{(2)} \end{bmatrix} \equiv \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - W_2 =$$

$$= \begin{bmatrix} 4 & 1 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.4 \\ 0.4 & 0.8 \end{bmatrix} = \begin{bmatrix} 3.8 & 0.6 \\ 1.6 & 4.2 \end{bmatrix}$$

c) Perform Step 1 on matrix A_2

$$3.8 = a_{11}^{(2)} = l_{11}^{(2)} u_{11}^{(2)} \Rightarrow u_{11}^{(2)} = 3.8$$

$$0.6 = a_{12}^{(2)} = l_{11}^{(2)} u_{12}^{(2)} \Rightarrow u_{12}^{(2)} = 0.6$$

$$1.6 = a_{21}^{(2)} = l_{21}^{(2)} u_{11}^{(2)} = 3.8\, l_{21} \Rightarrow l_{21}^{(2)} = 0.421$$

Situation after Step 2

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{11}^{(2)} & 0 \\ l_{31} & l_{21}^{(2)} & * \end{bmatrix} \quad ; \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{11}^{(2)} & u_{12}^{(2)} \\ 0 & 0 & * \end{bmatrix}$$

STEP 3

a) Form 1 × 1 matrix W_3

$$W_3 = l_{21}^{(2)} \cdot u_{23}^{(2)} = 0.2526$$

b) Form 1 × 1 matrix A_3

$$A_3 \equiv \left[\, a_{33}^{(3)} \,\right] = \left[\, a_{33}^{(2)} \,\right] - W_3 = 4.2 - 0.2526 = 3.9474$$

c) Perform Step 1 on matrix A_3

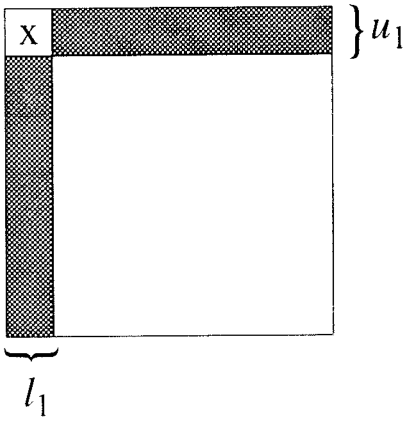
$$3.9474 = \left[\, a_{33}^{(3)} \,\right] = l_{11}^{(3)} u_{11}^{(3)} \quad \Rightarrow \quad u_{11}^{(3)} = 3.9474$$

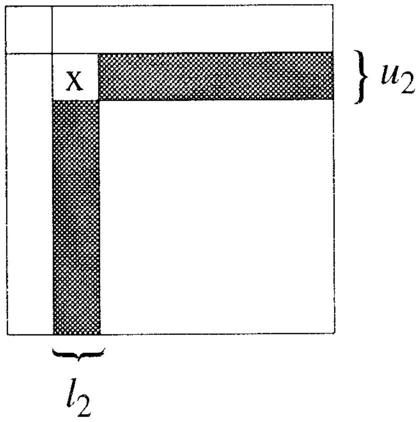
FINAL SITUATION

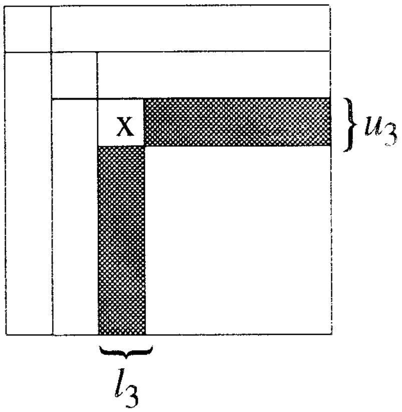
$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{11}^{(2)} & 0 \\ l_{31} & l_{21}^{(2)} & l_{11}^{(3)} \end{bmatrix} \quad ; \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{11}^{(2)} & u_{12}^{(2)} \\ 0 & 0 & u_{11}^{(3)} \end{bmatrix}$$

The matrices L and U are identical to those obtained using Algorithm 1.

COMMENT 1. Algorithm 2 computes L and U by recursively applying Step 1 to matrices A, A_1, A_2, ... In each step the dimension of the matrix to be factorized is reduced by 1.





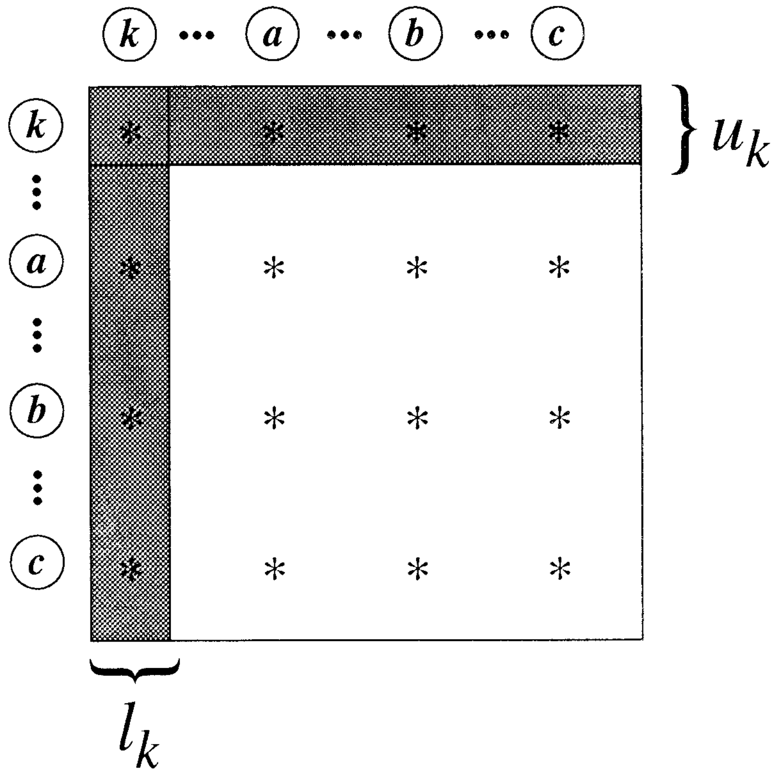


COMMENT 2. When executing Step 1 on matrix A_k, its nonzero pattern is *automatically replicated* in column k of L and row k of U (since $u_{1k} = a_{1k}/l_{11}$ and $l_{k1} = a_{k1}/u_{11}$).

COMMENT 3. When A is a *structurally symmetric* matrix, L and U have the *same* nonzero pattern.

COMMENT 4. When A is a *symmetric, positive definite* matrix, $U = L^T$. This special case is known as *Cholesky factorization.*

COMMENT 5. For a symmetric matrix A, the nonzero pattern of matrices L and U can be monitored and predicted *without actually computing these matrices.* For example, consider the computation of matrix W_{k+1}



If l_k has nonzeros only in rows a, b and c then W_{k+1} will have the following nine nonzero entries: (a, a), (a, b), (a, c); (b, a), (b, b), (b, c); (c, a), (c, b), (c, c). These nonzero elements must appear in A_{k+1} as well.

Why is it so important to predict the number and location of nonzeros in L and U ?

EXAMPLE

Matrix A (structurally symmetric)

$$
\begin{array}{c|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & * & * & * & * & * \\
2 & * & * & 0 & 0 & 0 \\
3 & * & 0 & * & 0 & 0 \\
4 & * & 0 & 0 & * & 0 \\
5 & * & 0 & 0 & 0 & *
\end{array}
$$

Matrices L and U (combined for convenience into one matrix)

$$
\begin{array}{c|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & * & * & * & * & * \\
2 & * & * & \odot & \odot & \odot \\
3 & * & \odot & * & \odot & \odot \\
4 & * & \odot & \odot & * & \odot \\
5 & * & \odot & \odot & \odot & *
\end{array}
$$

COMMENT 1. This example illustrates that even if A is sparse the corresponding matrices L and U can have many more nonzero elements, and the initial advantages of sparsity can be lost.

COMMENT 2.The additional nonzero elements that appear in L and U are referred to as *fill-ins*, and are denoted by ⊙. The *location* of these elements can be predicted (but not their numerical value).

Fill-in reduction

When matrices are large, fill-ins represent a critical problem. The amount of fill-in can be reduced by *permuting* the original matrix.

EXAMPLE (fill-in reduction by permutation)

If the matrix in the previous example is permuted as

$$\begin{array}{c|ccccc|}
 & 5 & 2 & 3 & 4 & 1 \\
\hline
5 & * & 0 & 0 & 0 & * \\
2 & 0 & * & 0 & 0 & * \\
3 & 0 & 0 & * & 0 & * \\
4 & 0 & 0 & 0 & * & * \\
1 & * & * & * & * & * \\
\end{array}$$

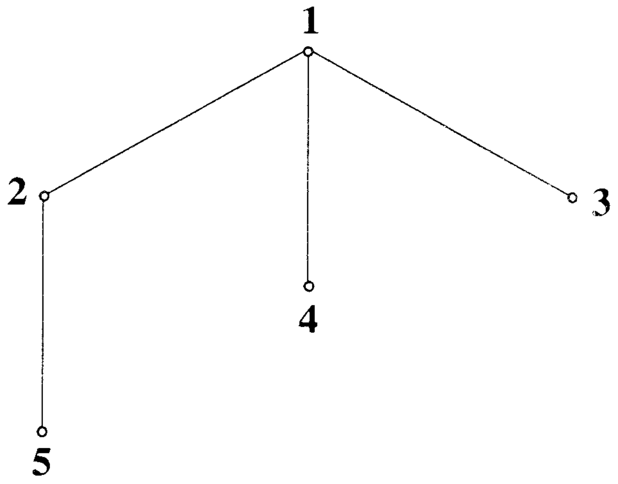
there is no fill-in at all!

With any structurally symmetric matrix A we can uniquely associate an undirected graph **G** in which vertices i and j are connected if and only if $a_{ji} \neq 0$ and $a_{ij} \neq 0$. In such a graph each vertex represents the corresponding matrix column, and edges represent nonzero elements.

EXAMPLE

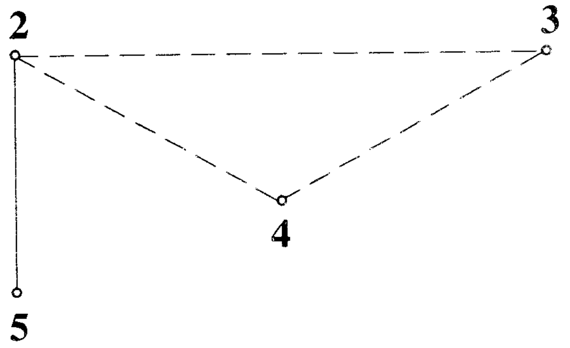
$$\begin{bmatrix} * & * & * & * & 0 \\ * & * & 0 & 0 & * \\ * & 0 & * & 0 & 0 \\ * & 0 & 0 & * & 0 \\ 0 & * & 0 & 0 & * \end{bmatrix}$$

Corresponding graph



ELIMINATION PROCEDURE. In each step, eliminate a vertex by *removing all edges incident to it.* All the neighbors of this vertex must now be *pairwise connected,* forming what is known as a *clique.* This procedure may require adding *new* edges to the graph; each such edge represents a new fill-in element.

Graph after removing vertex 1 (its neighbors are {2, 3, 4})

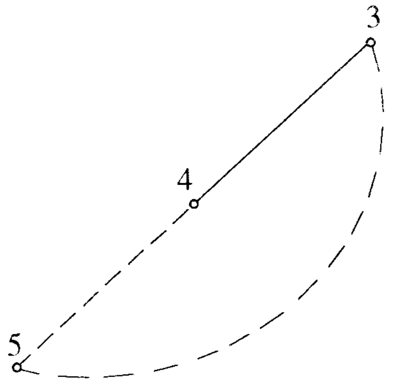


New fill-ins in this step: (3, 2); (4, 2); (4, 3)

Matrix A_2 (4 × 4)

$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} \ast & \ast & \ast & \ast & 0 \\ \ast & \ast & \odot & \odot & \ast \\ \ast & \odot & \ast & \odot & 0 \\ \ast & \odot & \odot & \ast & 0 \\ 0 & \ast & 0 & 0 & \ast \end{bmatrix} \end{array}$$

Graph after removing vertex 2 (its neighbors are {3, 4, 5})

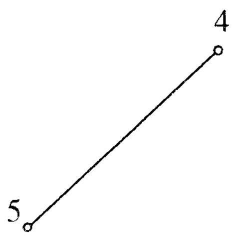


New fill-ins in this step: (5, 3) and (5, 4).

Matrix A_3 (3 × 3)

$$\begin{array}{c|cccc|} & 2 & 3 & 4 & 5 \\ 2 & * & * & * & * \\ 3 & * & * & * & \odot \\ 4 & * & * & * & \odot \\ 5 & * & \odot & \odot & * \end{array}$$

Graph after removing vertex 3 (its neighbors are {4, 5})



New fill-ins in this step: none.

Matrix A_4 (2×2)

$$\begin{array}{c|ccc|} & 3 & 4 & 5 \\ \hline 3 & * & * & * \\ 4 & * & * & * \\ 5 & * & * & * \end{array}$$

Graph after removing vertex 4 (its neighbor is $\{5\}$)

New fill-ins in this step: none.

Matrix A_5 (1×1)

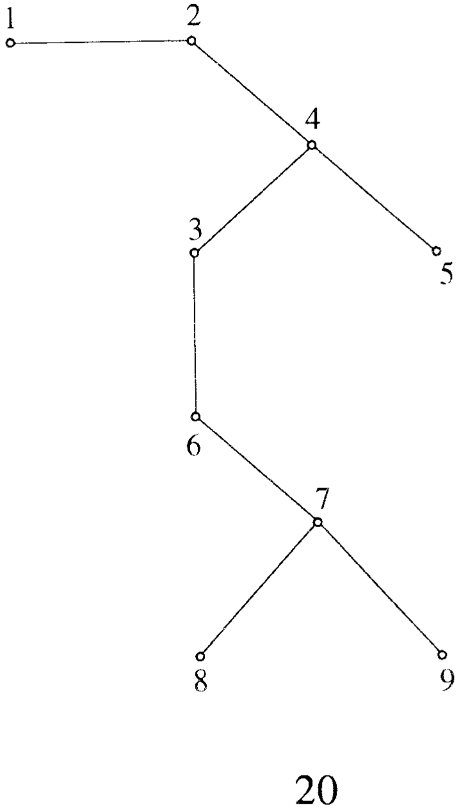
$$\begin{array}{c|cc|} & 4 & 5 \\ \hline 4 & * & * \\ 5 & * & * \end{array}$$

FINAL NON-ZERO PATTERN OF L

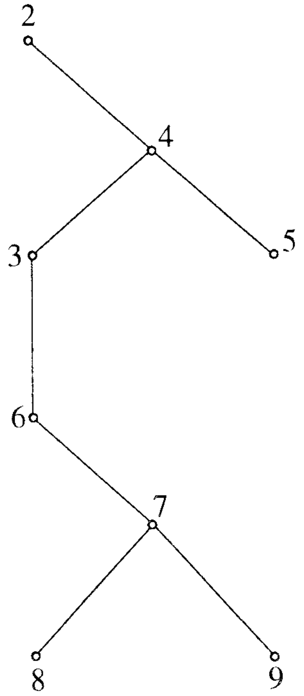
$$L = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \left[\begin{array}{ccccc} * & & & & \\ * & * & & & \\ * & \odot & * & & \\ * & \odot & \odot & * & \\ 0 & * & \odot & \odot & * \end{array} \right]$$

COMMENT. This example illustrates that the non-zero pattern of L and U can be monitored *directly* from the elimination graph, bypassing the explicit construction of matrices A_2, A_3, ...

EXAMPLE

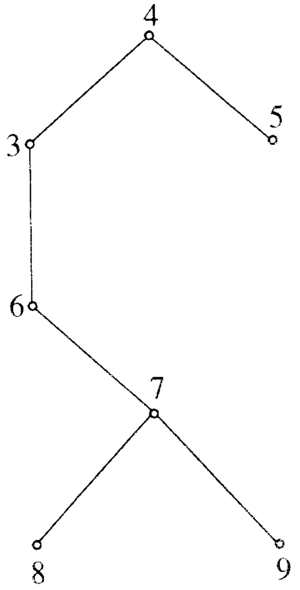


Graph after removing vertex 1 (its neighbor is {2})



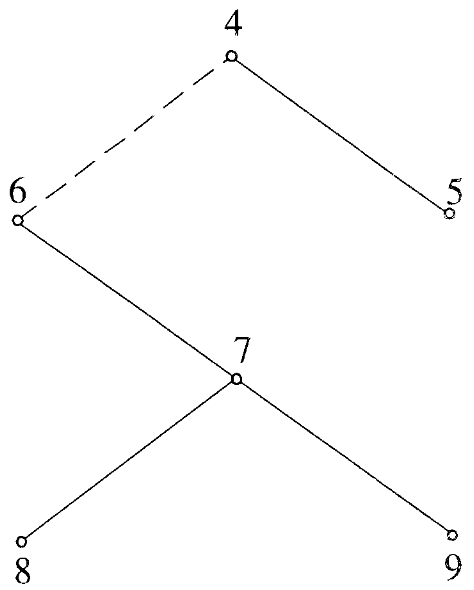
New fill-ins in this step: none.

Graph after removing vertex 2 (its neighbor is {4})



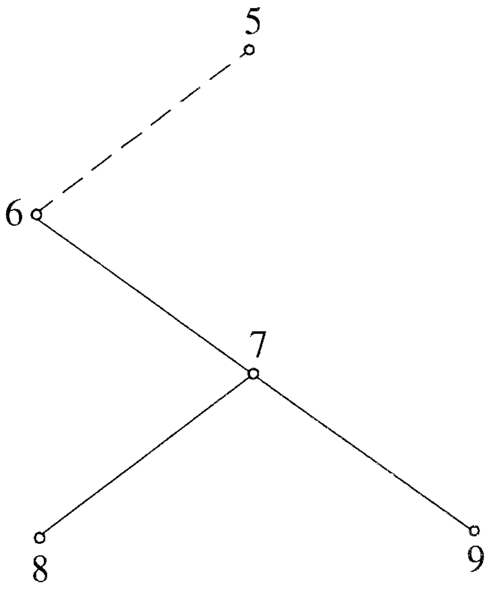
New fill-ins in this step: none.

Graph after removing vertex 3 (its neighbors are {4, 6})



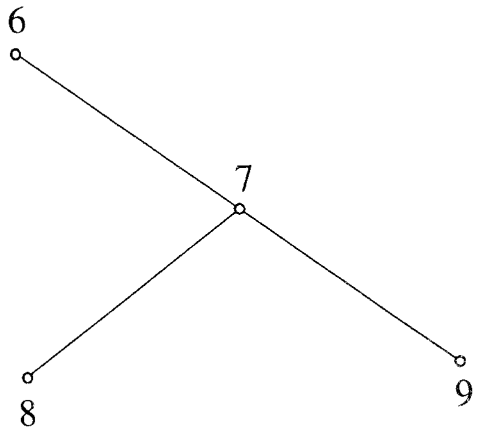
New fill-ins in this step: (6, 4).

Graph after removing vertex 4 (its neighbors are {5, 6})



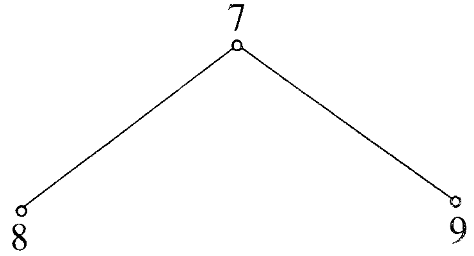
New fill-ins in this step: (6, 5).

Graph after removing vertex 5 (its neighbor is {6})

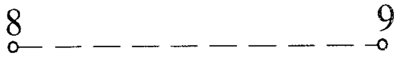


New fill-ins in this step: none.

Graph after removing vertex 6 (its neighbor is {7})

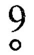


Graph after removing vertex 7 (its neighbors are {8, 9})



New fill-ins in this step: (9, 8).

Graph after removing vertex 8 (its neighbor is {9})



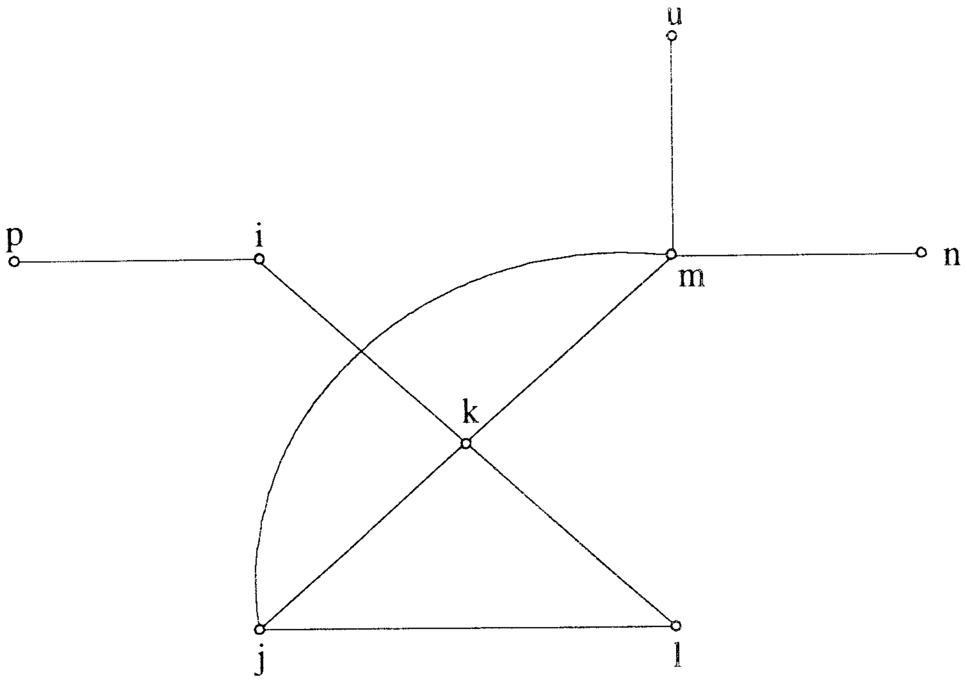
TOTAL FILL-IN IN L: (6, 4); (6, 5) and (9, 8)

Use of cliques for storage enhancement

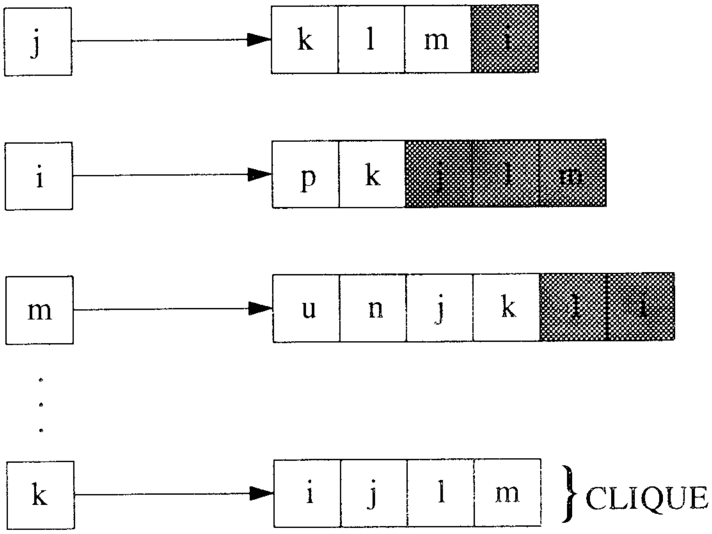
Whenever a fill-in occurs, the non-zero pattern changes and an additional element needs to be stored. In general, the added storage requirements can be very significant.

EXAMPLE

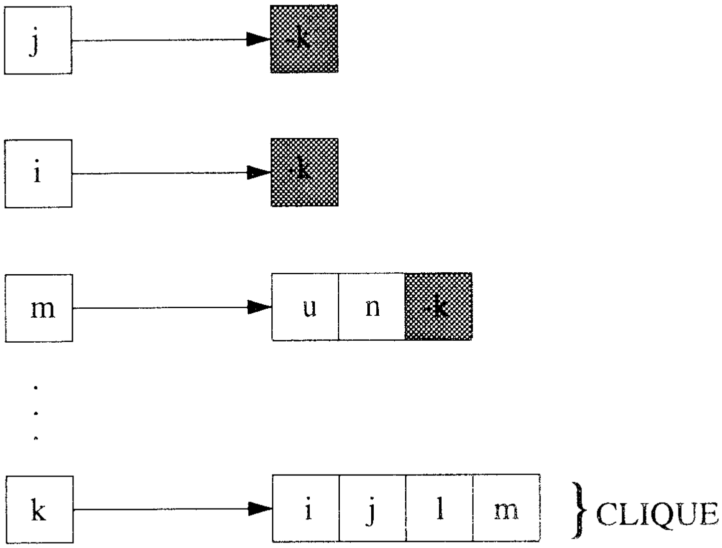
Suppose that we want to remove vertex *k* from the elimination graph below (*k* has neighbors {*i*, *j*, *l*, *m*})



New fill-in elements are (i, j); (i, m); (i, l) and (l, m). The data structure from the previous step is substantially enlarged (added elements are shaded)



We know that the elimination of vertex k creates a *clique* $\{k, i, j, l, m\}$, which all become pairwise connected. Therefore, instead of adding them to the data structure, we can replace the *whole clique* by - k (so-called storage by reference)



This can result in a major reduction in storage space.

Minimizing the amount of fill-in is a very difficult problem (so-called *NP*-complete). As a result, all algorithms of this type are *heuristic.*

MINIMAL DEGREE ORDERING

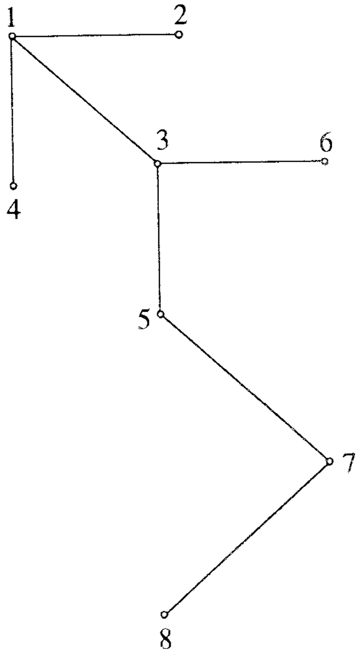
A very effective general purpose algorithm, which is included in practically all sparse matrix software packages.

PROCEDURE. Form the elimination graph, but do not eliminate vertices in sequence. Instead, *in each step eliminate the vertex with the minimal degree in the graph.*

EXAMPLE

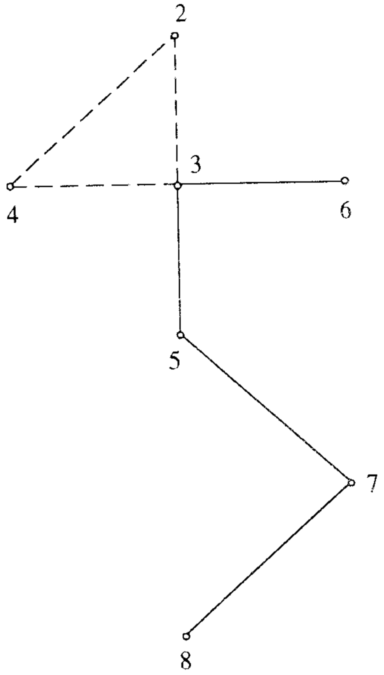
$$\begin{array}{c|cccccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
1 & * & * & * & * & 0 & 0 & 0 & 0 \\
2 & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & * & 0 & * & 0 & * & * & 0 & 0 \\
4 & * & 0 & 0 & * & 0 & 0 & 0 & 0 \\
5 & 0 & 0 & * & 0 & * & 0 & * & 0 \\
6 & 0 & 0 & * & 0 & 0 & * & 0 & 0 \\
7 & 0 & 0 & 0 & 0 & * & 0 & * & * \\
8 & 0 & 0 & 0 & 0 & 0 & 0 & * & *
\end{array}$$

Graph **G**



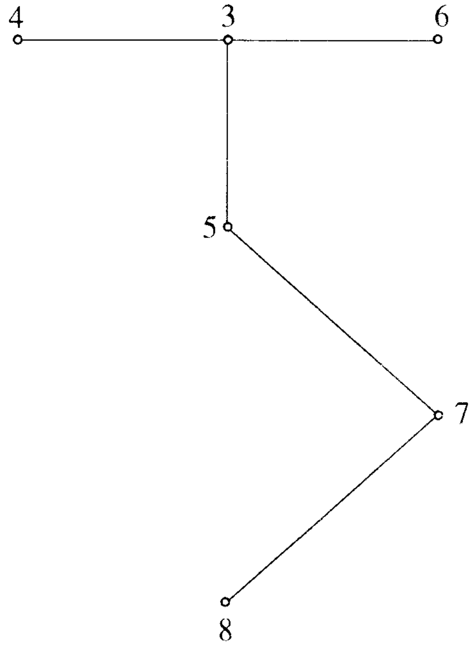
WITHOUT ORDERING

Graph after removing vertex 1 (its neighbors are {2, 3, 4})



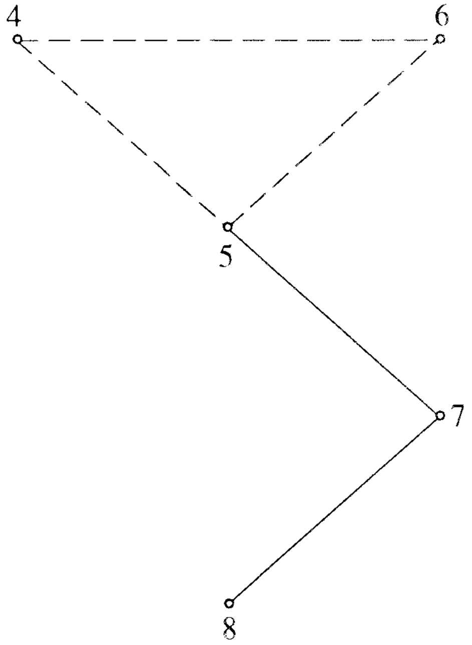
New fill-ins in this step: (3, 2), (4, 2), (4, 3).

Graph after removing vertex 2 (its neighbors are {3, 4})



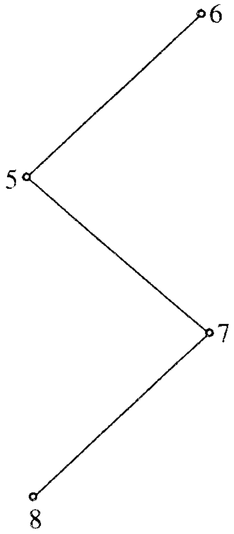
New fill-ins in this step: none.

Graph after removing vertex 3 (its neighbors are {4, 5, 6})



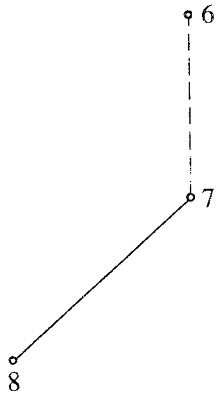
New fill-ins in this step: (5, 4), (6, 4), (6, 5).

Graph after removing vertex 4 (its neighbors are {5, 6})



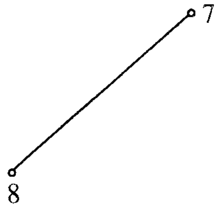
New fill-ins in this step: none.

Graph after removing vertex 5 (its neighbors are {6, 7})



New fill-ins in this step: (7, 6).

Graph after removing vertex 6 (its neighbor is {7})



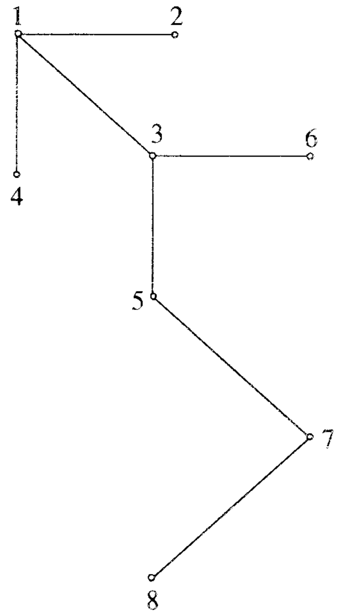
New fill-ins in this step: none.

THE FINAL NONZERO STRUCTURE IN *L* AND *U* (14 fill-ins)

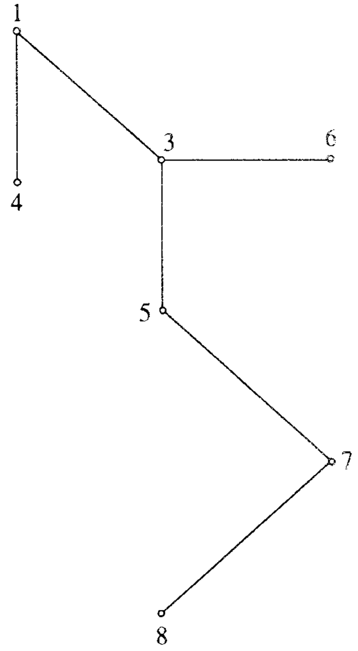
$$\begin{array}{cc}
 & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} &
\left[\begin{array}{cccccccc}
\ast & \ast & \ast & \ast & 0 & 0 & 0 & 0 \\
\ast & \ast & \odot & \odot & 0 & 0 & 0 & 0 \\
\ast & \odot & \ast & \odot & \ast & \ast & 0 & 0 \\
\ast & \odot & \odot & \ast & \odot & \odot & 0 & 0 \\
0 & 0 & \ast & \odot & \ast & \odot & \ast & 0 \\
0 & 0 & \ast & \odot & \odot & \ast & \odot & 0 \\
0 & 0 & 0 & 0 & \ast & \odot & \ast & \ast \\
0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast
\end{array}\right]
\end{array}$$

WITH MINIMAL DEGREE ORDERING

Graph **G**

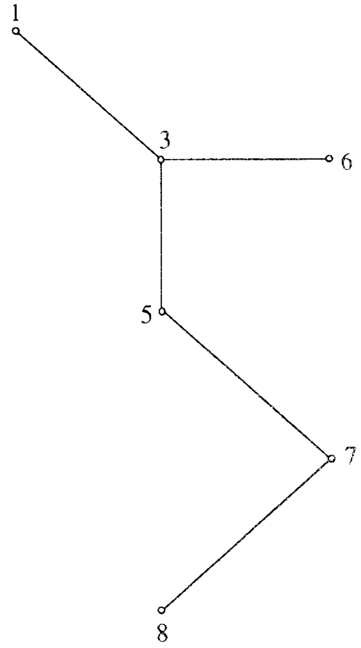


Graph after removing vertex 2 (its neighbor is {1})



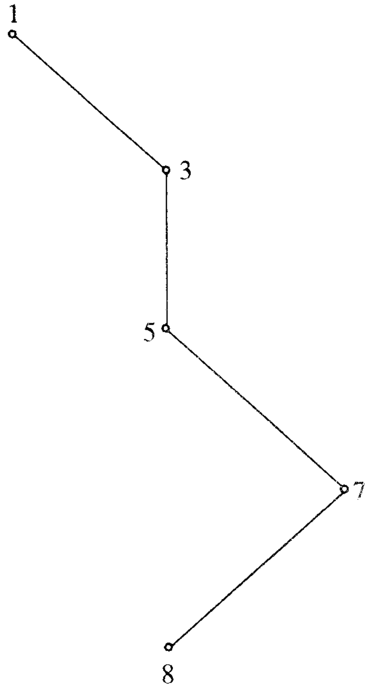
New fill-ins in this step: none.

Graph after removing vertex 4 (its neighbor is {1})



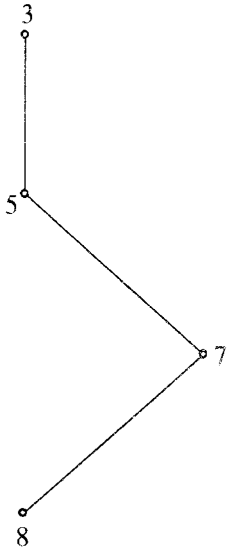
New fill-ins in this step: none.

Graph after removing vertex 6 (its neighbor is $\{3\}$)



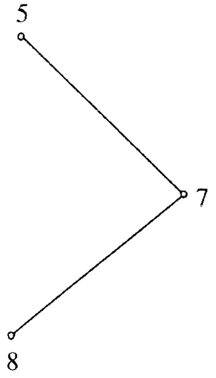
New fill-ins in this step: none.

Graph after removing vertex 1 (its neighbor is $\{3\}$)



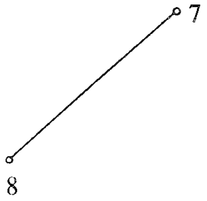
New fill-ins in this step: none.

Graph after removing vertex 3 (its neighbor is $\{5\}$)



New fill-ins in this step: none.

Graph after removing vertex 5 (its neighbor is $\{7\}$)



New fill-ins in this step: none.

THE FINAL NONZERO STRUCTURE IN L AND U (no fill-ins)

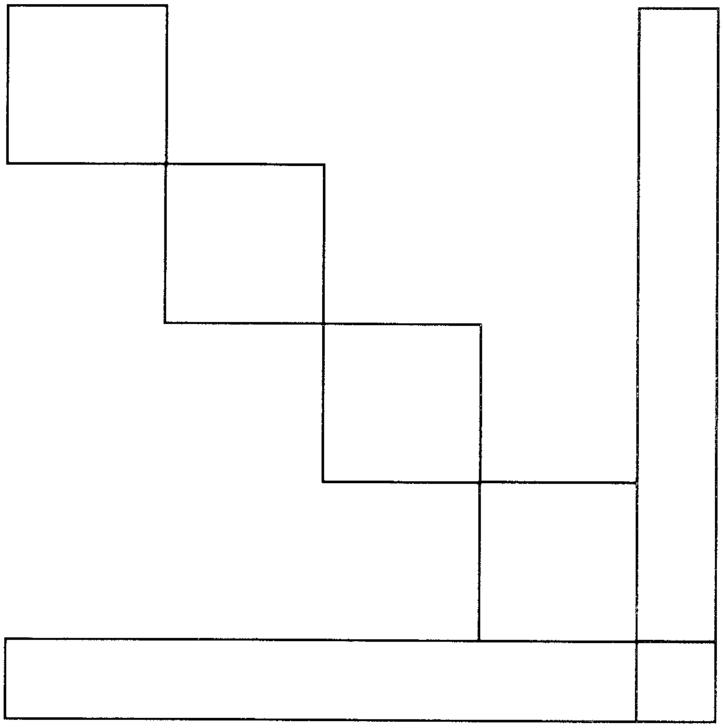
$$
\begin{array}{c|cccccccc|}
 & 2 & 4 & 6 & 1 & 3 & 5 & 7 & 8 \\
\hline
2 & * & 0 & 0 & * & 0 & 0 & 0 & 0 \\
4 & 0 & * & 0 & * & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & * & 0 & * & 0 & 0 & 0 \\
1 & * & * & 0 & * & * & 0 & 0 & 0 \\
3 & 0 & 0 & * & * & * & * & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & * & * & * & 0 \\
7 & 0 & 0 & 0 & 0 & 0 & * & * & * \\
8 & 0 & 0 & 0 & 0 & 0 & 0 & * & *
\end{array}
$$

COMMENT 1. Minimal degree is very effective in reducing the amount of fill-in. However, the reordered matrix lacks structure, and it can be difficult to implement LU factorization in a multiprocessor environment.

COMMENT 2. When several vertices in the elimination graph have the same degree, how do we decide which is the next to be eliminated? Different tie-breaking criteria give rise to different variations of the minimal degree ordering. A good tie-breaking scheme can further reduce the amount of fill-in.

THE BORDERED BLOCK DIAGONAL STRUCTURE

This type of matrix structure is well suited for parallel processing. The typical format is



There are many algorithms that attempt to achieve this structure and simultaneously minimize the amount of fill-in.

Nested Dissection

Very successful for matrices that already have some regularity in their structure. However, much less successful in matrices with irregular patterns (such as those arising in circuits).

SOME PRELIMINARY DEFINITIONS

DEFINITION. The *eccentricity* of vertex x in a graph, denoted $l(x)$, is the maximal distance from x to any other vertex in the graph

$$l(x) = \max_{y \in G} d(x, y)$$

DEFINITION. The *diameter* of graph **G**, denoted $\delta(\mathbf{G})$, is the maximal distance between any two vertices in **G**

$$\delta(G) \equiv \max_{x \in G} l(x)$$

DEFINITION. Vertex x is said to be a *peripheral* vertex in graph **G** if $l(x) = \delta(\mathbf{G})$. A vertex that is nearly peripheral will be referred to as a *pseudo-periferal* vertex.

PROCEDURE

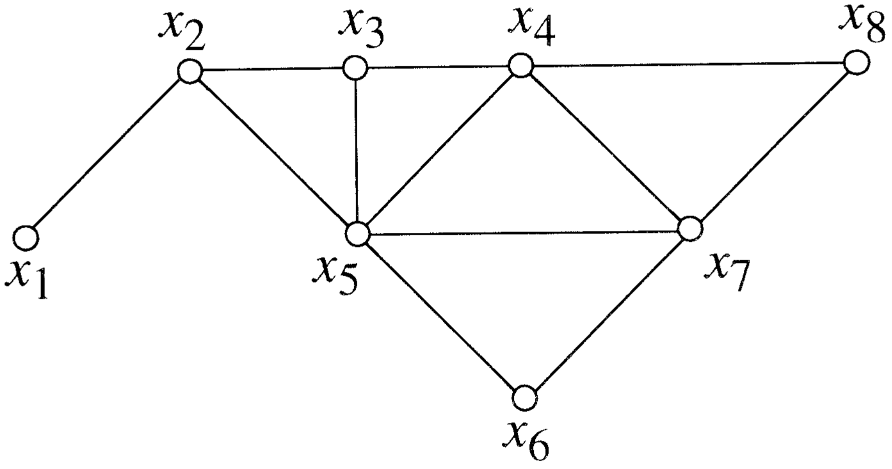
(*i*) Find a pseudo-peripheral vertex, and generate the corresponding *rooted level structure.*

(*ii*) Identify the *middle* level in this structure. All the vertices in this level set now become candidates for a *separator*, whose removal will break up the graph into two disconnected components. It is actually necessary to remove only vertices that are connected to the next level, so the separator is normally smaller than the middle level set.

(*iii*) After removing the separator, repeat the first two steps on the remaining components of the graph until some assigned criterion is satisfied.

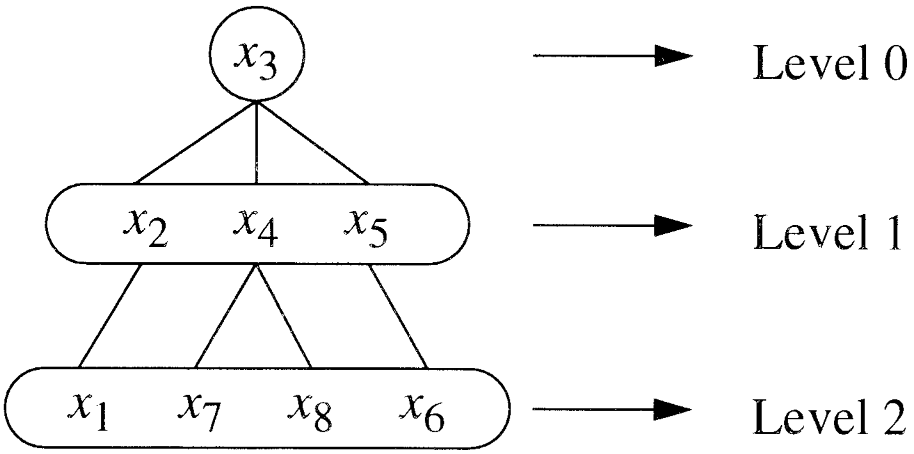
EXAMPLE

Consider the following graph

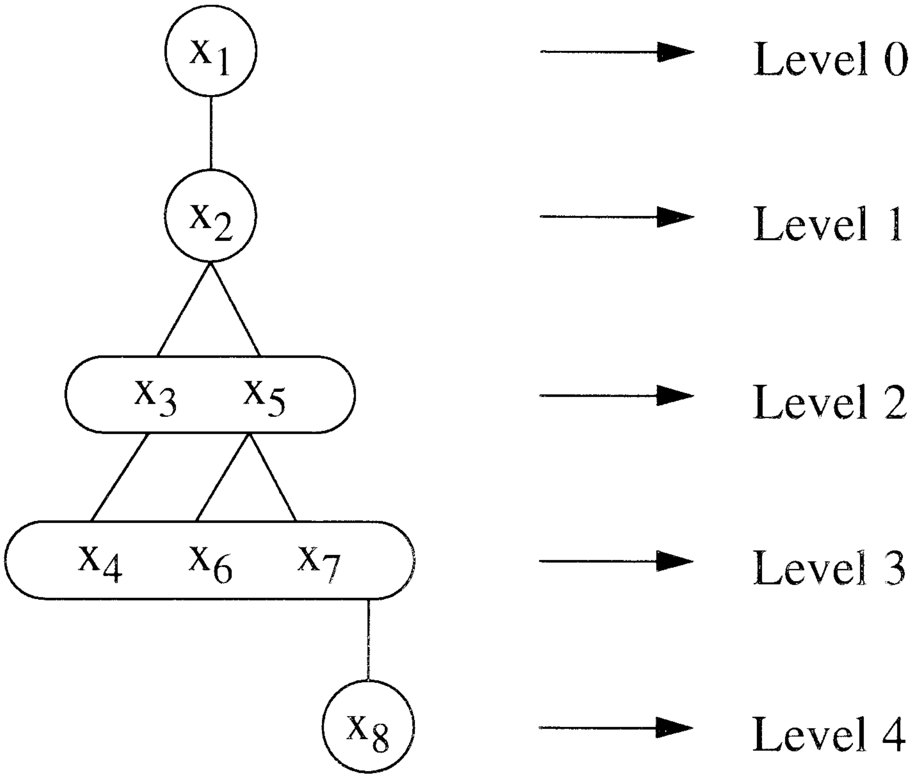


Our first step will be to find a pseudo-peripheral vertex.

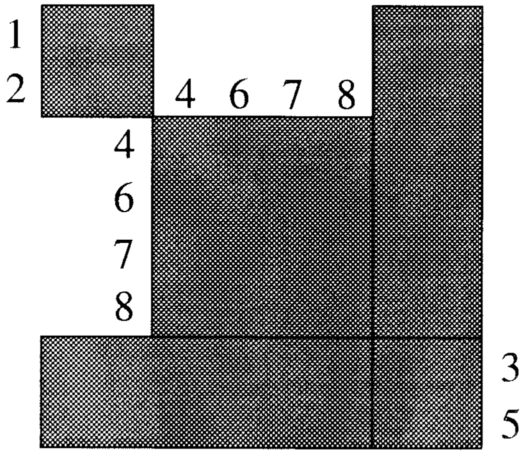
Selecting vertex x_3 as the root, we obtain the following rooted level structure



Since there are only two levels, select the vertex from the *last* level with the *smallest degree* (in this case, x_1 is such a vertex).

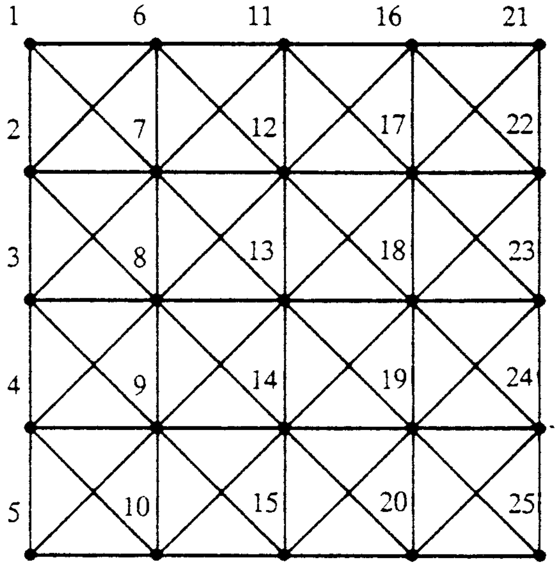


Repeating the procedure in this case will not increase the number of levels. Therefore x_1 is a *pseudo-peripheral* node, and $\{x_3, x_5\}$ represent the *minimal separator.* The resulting BBD structure is



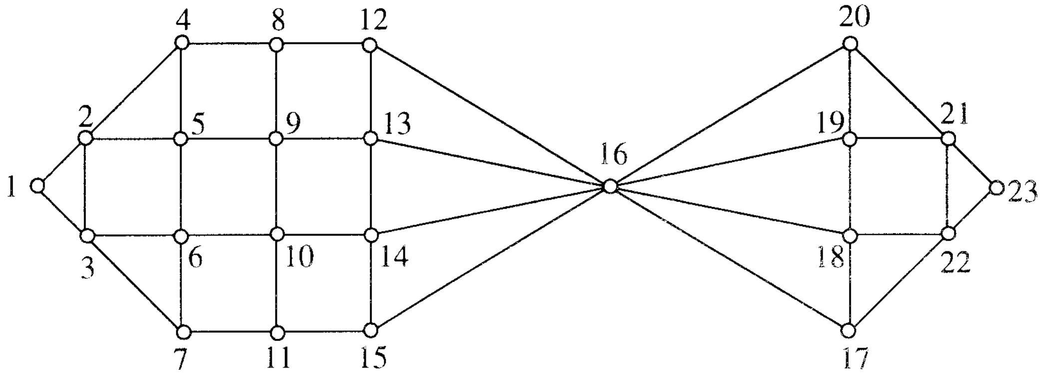
COMMENT 1. This example illustrates that the separator in fact represents the *border* of the BBD matrix.

COMMENT 2. A common approach to reducing fill-in has been to minimize the size of the border. By this criterion, nested dissection does well for regular matrices, such as the one corresponding to the graph below

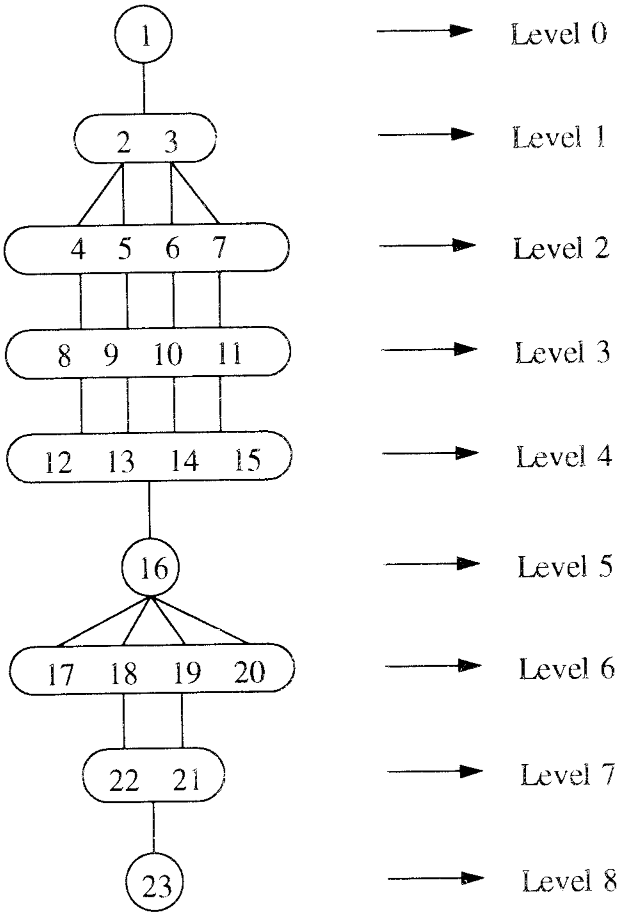


EXAMPLE

Consider the following irregular graph



The level structure rooted at vertex 1 is



Following the nested dissection algorithm, the minimal separator is the set {12, 13, 14, 15}. However, by inspection it follows that {16} is a much better choice. Consequently, in this case nested dissection does not to too well.

Decompositions based on eigenvectors of graphs

DEFINITION. Let **G** be an undirected graph, in which V and E denote the set of vertices and edges, respectively. The *adjacency matrix* A is then defined by: $a_{ij} = 1$ if $(i, j) \in E$ and $a_{ij} = 0$ otherwise. By definition, $a_{ii} = 0\ ,\ \forall i$.

DEFINITION. Let $d(v)$ denote the degree of vertex v, and define a diagonal matrix

$$D = \text{diag}\{d(1), d(2), \ldots, d(n)\}$$

The matrix Q defined as $Q \equiv D - A$ will now be referred to as the *Laplacian matrix* of graph **G**.

COMMENT. Matrix Q is always positive semi-definite, with at least one zero eigenvalue. The smallest positive eigenvalue of Q is denoted λ_2, and the corresponding eigenvector is denoted by X_2.

PROCEDURE

(*i*) Compute eigenvector X_2, and determine its *median* component x_l.

(*ii*) Partition the vertices of the graph in the following way: for any vertex *i*, if $x_i < x_l$, set $i \in$ A; otherwise, $i \in$ B. In this way, the vertices of **G** will be partitioned into two approximately equal sets, A and B.

(*iii*) All the edges connecting sets A and B now constitute an *edge separator* H. The objective now is to find a minimal *vertex cover* for H (that is, the minimal number of vertices that need to be removed so that all edges $\in$ H are removed). This vertex cover constitutes the *separator*.

(*iv*) Repeat steps (*i*) - (*iii*) on the remaining components after the separator is removed.

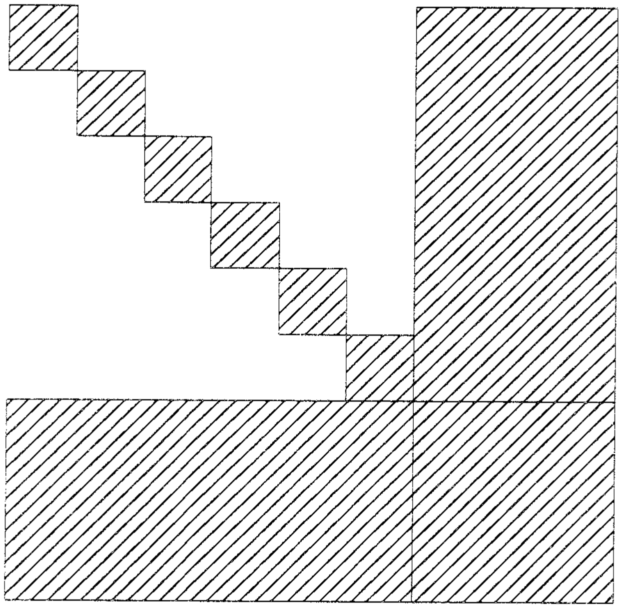
COMMENT. This algorithm performs well for both regular and irregular matrix structures. However, that for large matrices computing the second eigenvector can be very difficult, if not impossible .

Balanced BBD decompositions

This is an algorithm that we developed at Santa Clara University. It is primarily designed for parallel computation, and has several features that give it an advantage over algorithms such as nested dissection or graph eigenvectors.

The algorithm is recursive and has two basic steps.

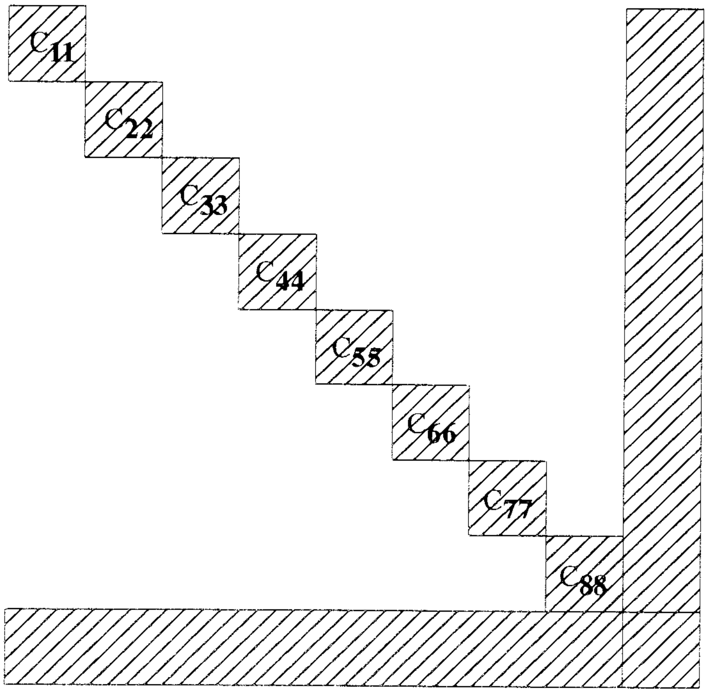
STEP 1. Select a *maximal allowable block size* Nmax. Given this choice, move as many vertices as necessary to the border so that each block has size ≤ Nmax. A typical situation after this step is



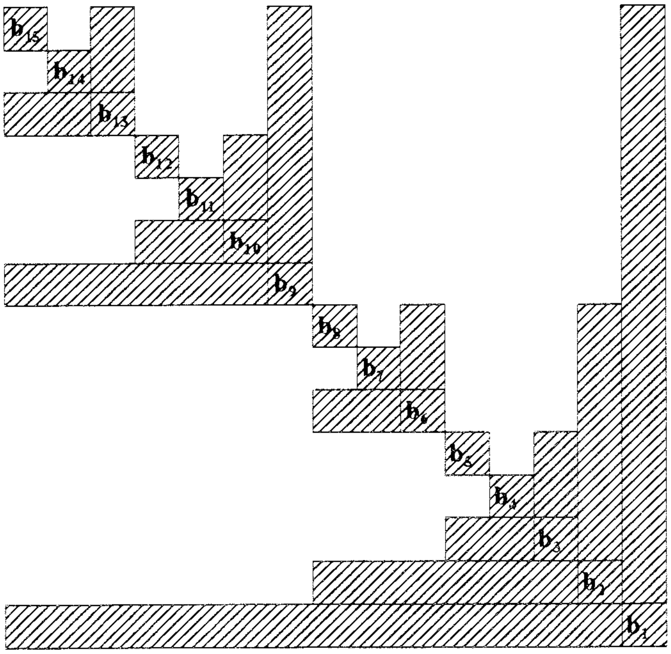
STEP 2. The border is obviously too large after the first step; consequently, in step 2 we reconnect border vertices one by one. In this process, the next vertex to be reconnected is always the one that results in the smallest increase in block sizes (we call this a "greedy" algorithm). The process continues as long as we have at least two blocks left (in other words, we will stop when we see that the next reconnection will result in a single block).

Once we have two blocks and an initial border, steps 1 and 2 are repeated on *each* block (this makes the algorithm nested). The local borders are then moved and "attached" to the initial border. We continue with this procedure recursively until the border and all the diagonal blocks are approximately the same size (i. e. "balanced").

A typical structure resulting from this decomposition is shown below.



Note also that the border will have an internal structure, which is *preserved* in the process of LU factorization.



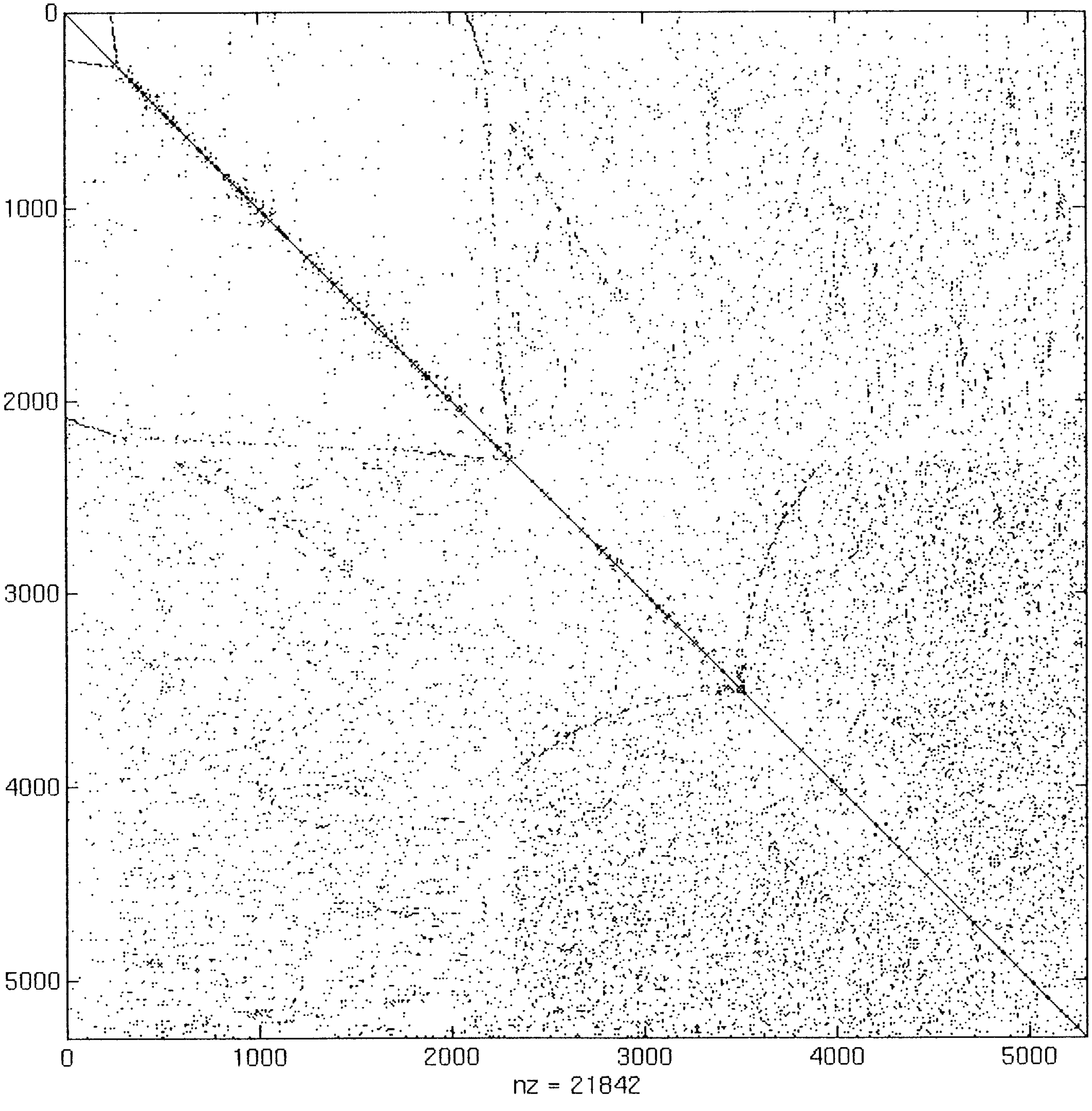
ADVANTAGES OF BALANCED BBD DECOMPOSITION

1) All diagonal blocks are of *similar size.* As a result, in parallel computations the work load is well balanced accross the processors (this is of fundamental importance).

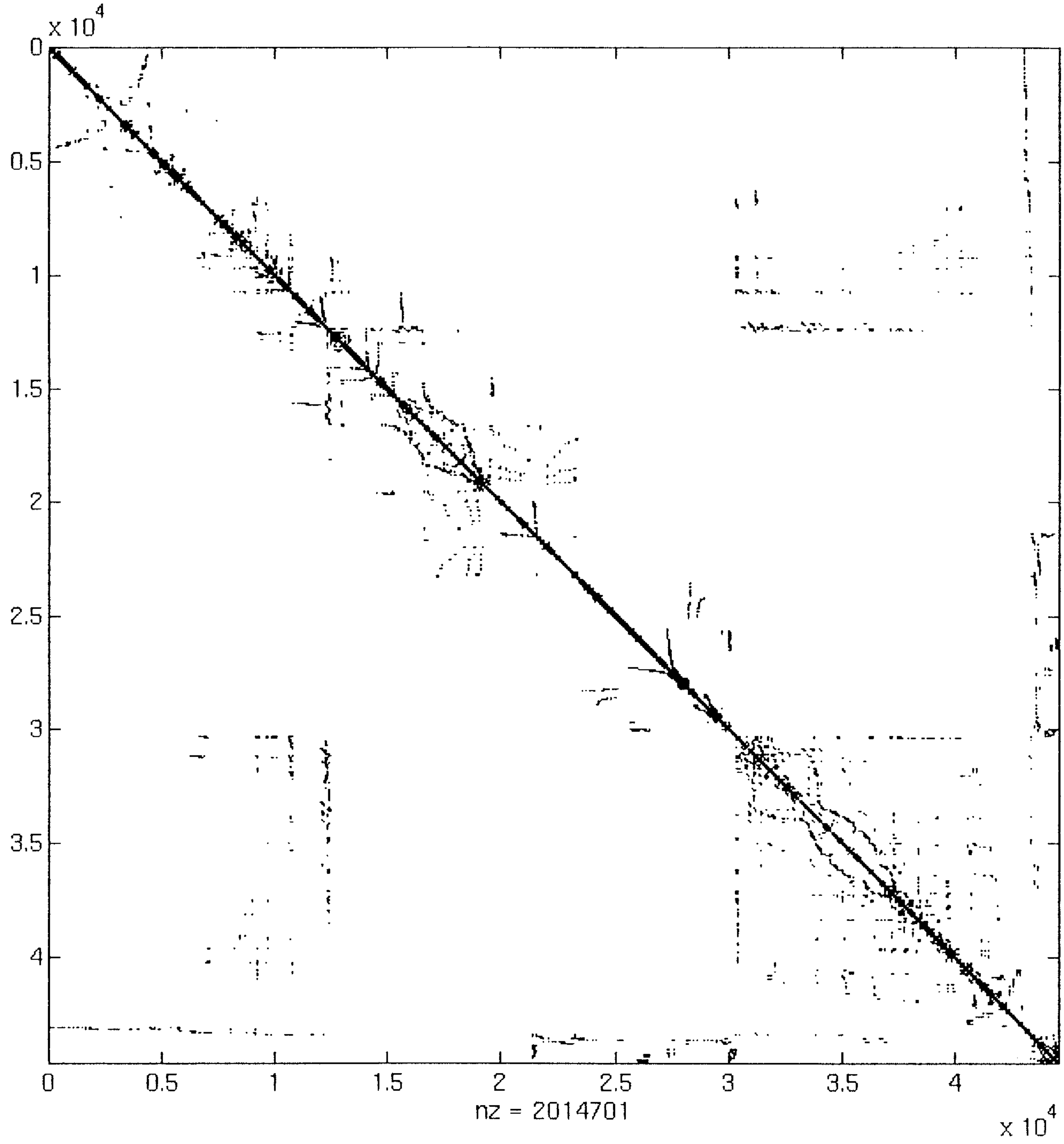
2) The algorithm is numerically simple and fast, because we only care about the *size* of a block and not its contents (unlike minimal degree and other orderings). In very large matrices, our algorithm is typically 4 times faster than the minimal degree ordering.

3) The amount of fill in is similar as in the case of minimal degree ordering. However, unlike minimal degree, we also get a structure that is perfectly suited for parallel computing.

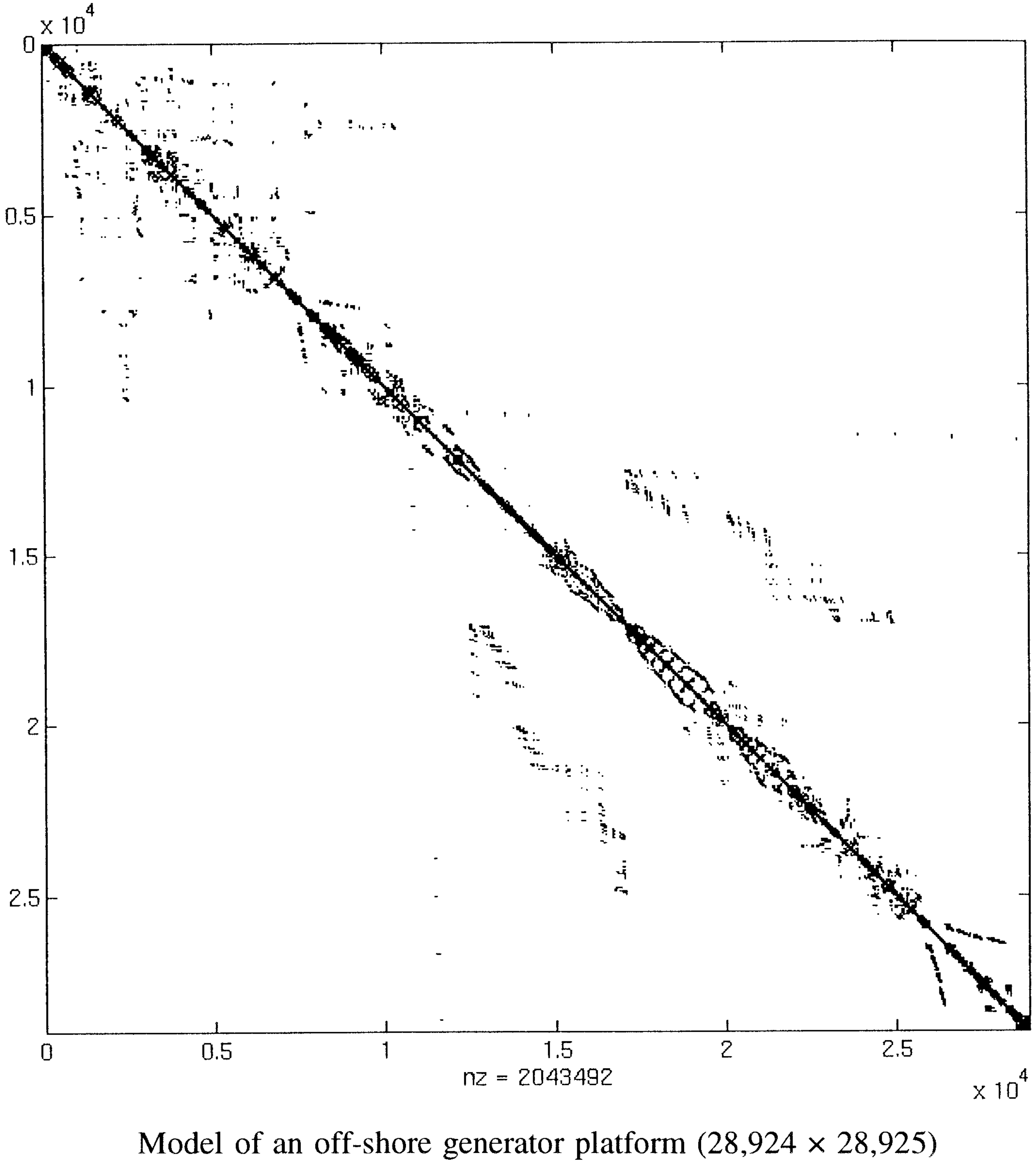
4) The balanced BBD decomposition works well for *all* types of matrices. This is unlike nested dissection, which gives good results for regular structures but does poorly for irregular matrices (such as those arising in circuits).



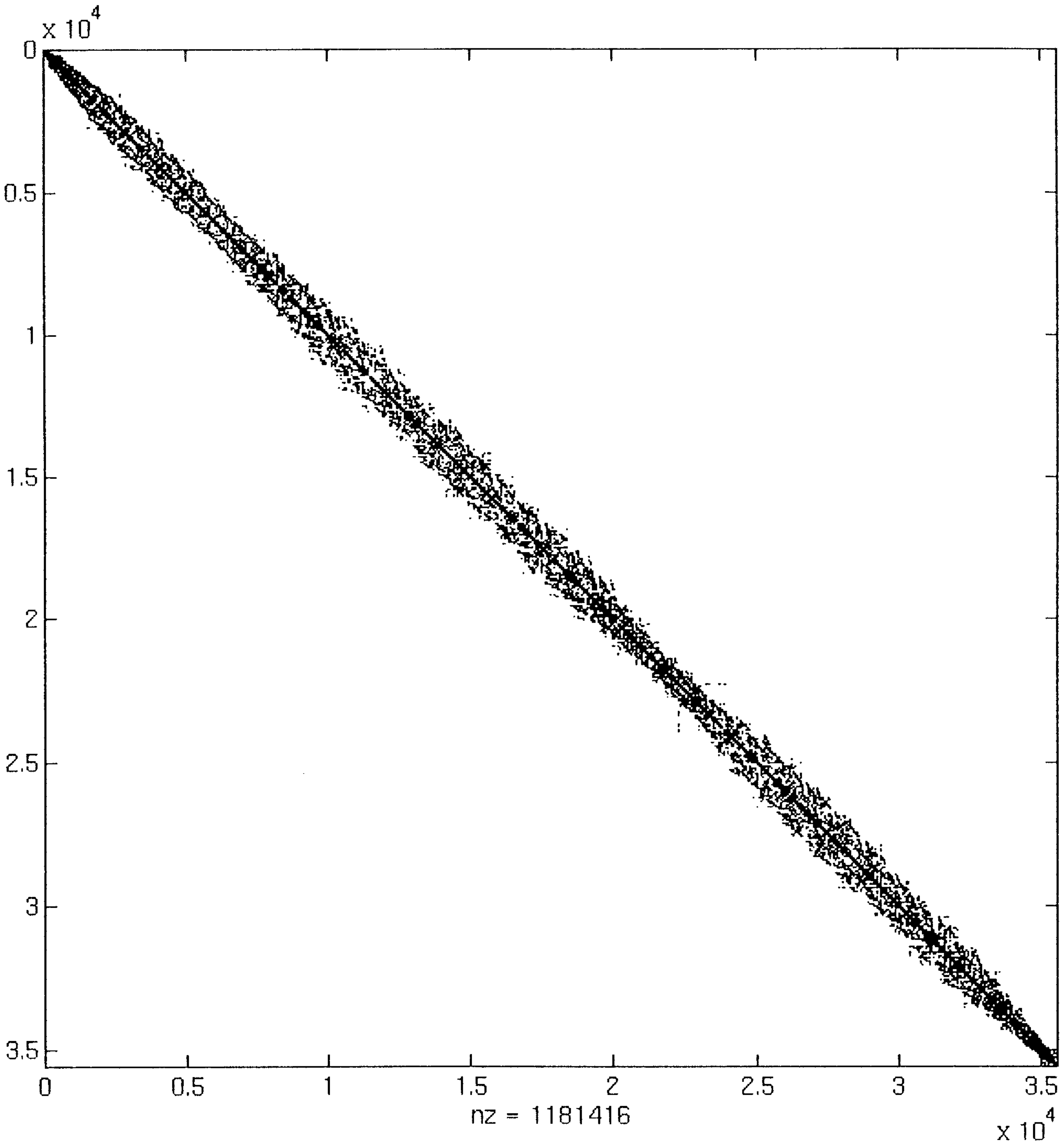
Model of the U. S. power network (5,300 × 5,300)



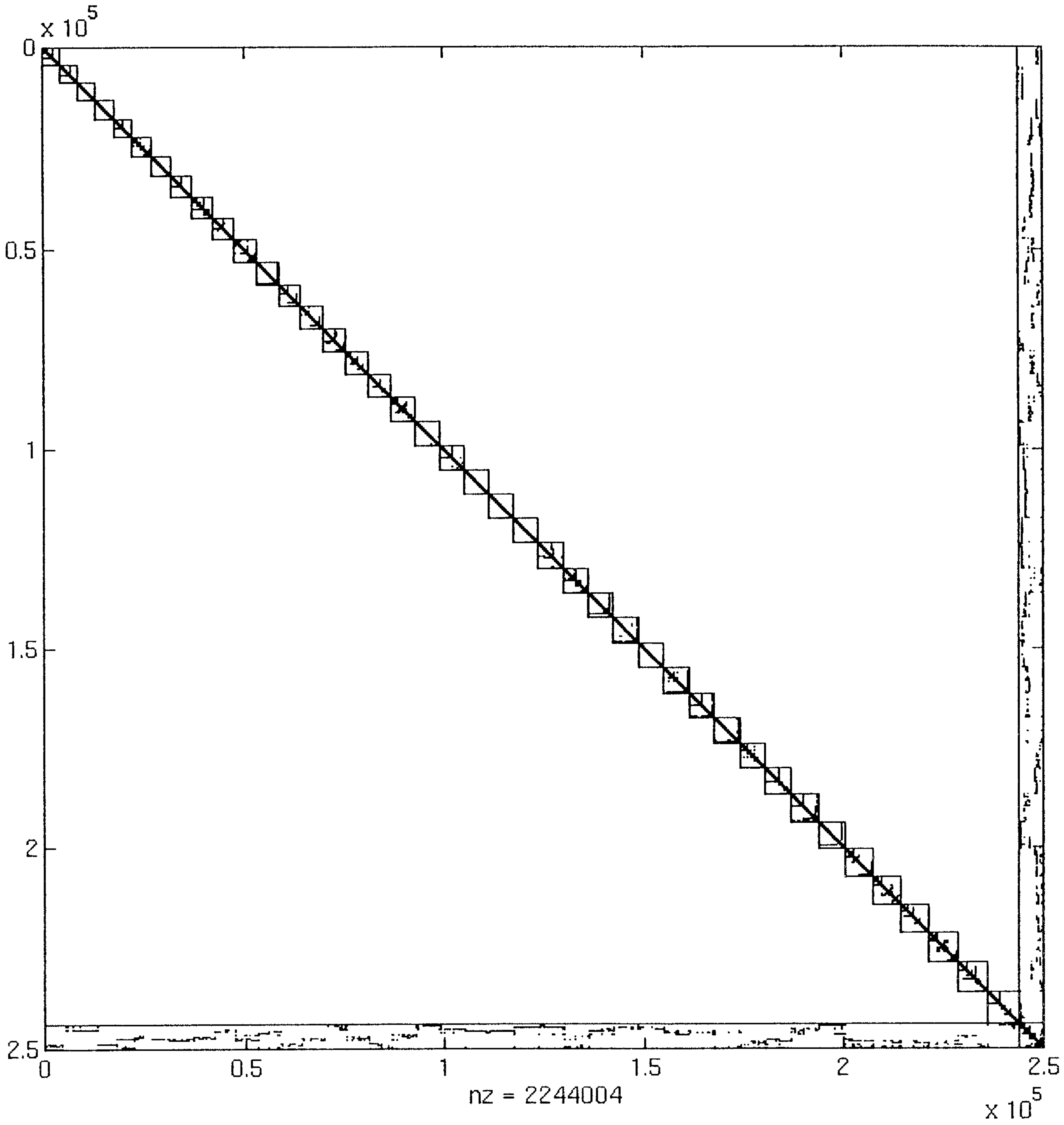
Model of an automobile chassis (46,609 × 46,609)



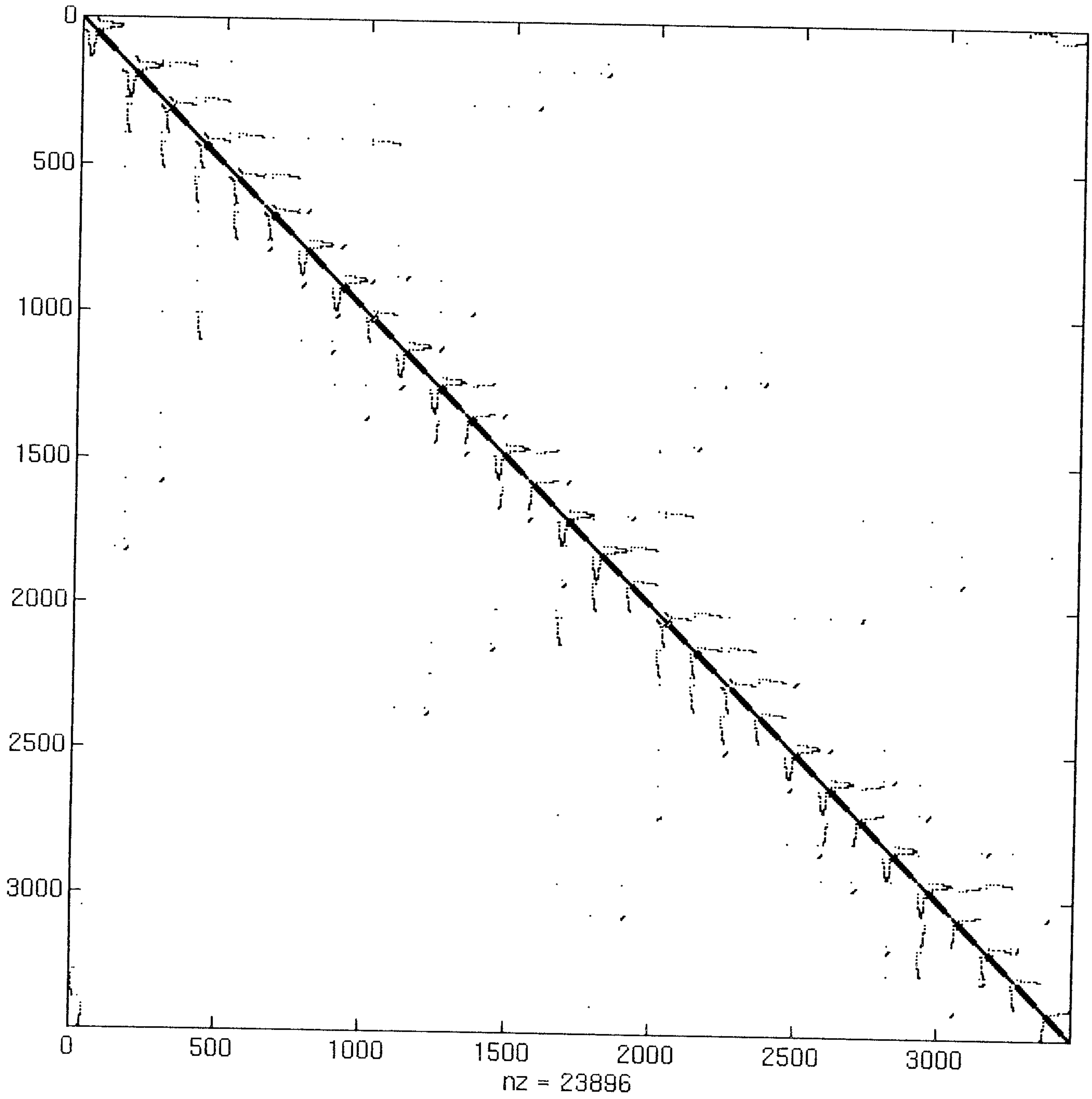
Model of an off-shore generator platform (28,924 × 28,925)



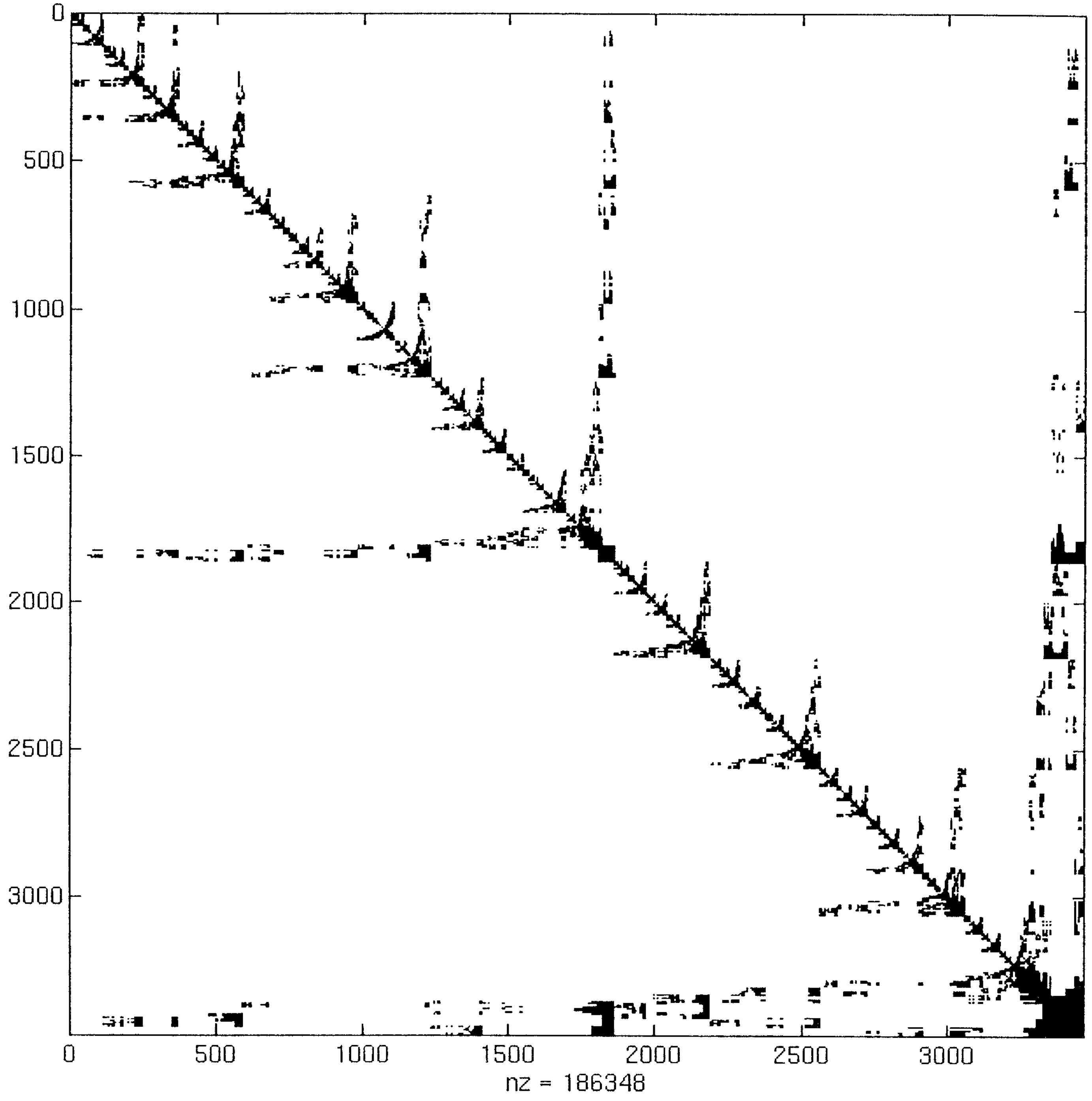
Model of an automobile steering component (35,588 × 35,588)



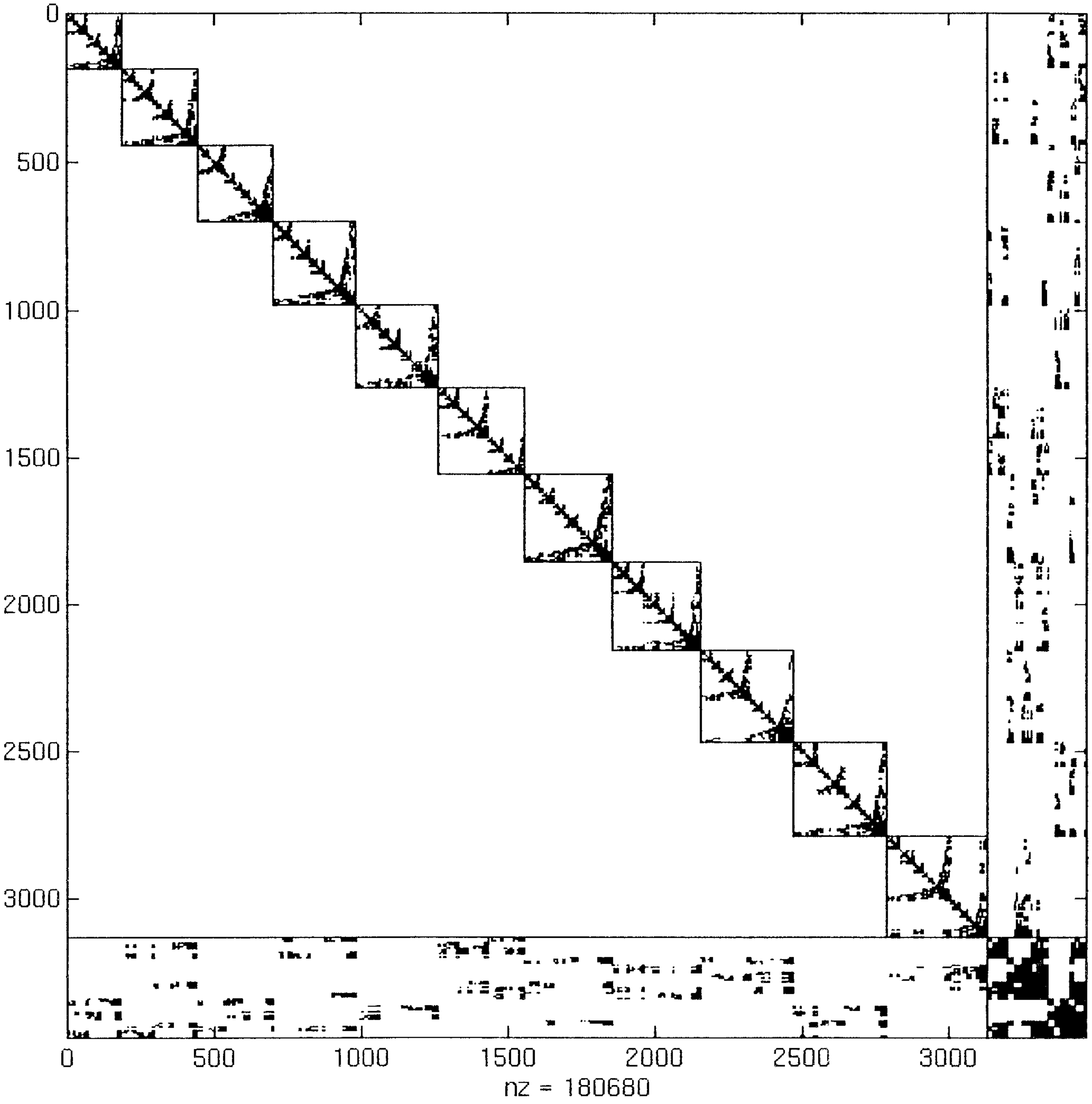
Balanced BBD decomposition of a nine point finite element model on a 500 × 500 grid (matrix size is 250,00 × 250,000)



An L-shaped heat conduction problem (3,466 × 3,466)



LU factors after a minimal degree ordering



LU factors after a balanced BBD decomposition

Matrix size	**Number of nonzeros**	**Ordering time**		**Tm/Tb**
		Min. deg. (Tm)	BBD (Tb)	
2,003	83,883	14.47 s	1.86 s	7.78
8,738	591,904	101.93 s	91.79 s	1.11
10,974	428,650	120.69 s	31.79 s	3.8
11,948	149,090	38.26 s	42.87 s	0.89
13,992	619,488	129.08 s	67.23 s	1.92
28,924	2,043,492	1,118.2 s	287.15 s	3.9
35,588	1,181,416	434.02 s	105.54 s	4.11
44,609	2,104,701	1,031.9 s	241.92 s	4.26
90,000	806,404	33.31 s	31.08 s	1.07
250,000	2,244,004	96.29 s	47 s	2.05

Table 1. A comparison of execution times for symmetric minimal degree and BBD orderings.

Matrix size	**Number of nonzeros**	**Nonzeros in L and L^T**		**Nm/Nb**
		Min. deg. (Nm)	BBD (Nb)	
1,005	8,621	41,555	36,801	1.13
1,074	12,960	72,426	74,494	0.97
1,084	3,966	6,426	6,786	0.95
1,143	18,552	23,452	22,022	1.06
1,806	63,454	238,868	258,304	0.92
1,993	7,443	13,439	14,259	0.94
2,003	83,883	588,887	568,545	1.04
3,466	23,896	186,348	180,680	1.03
4,884	290,365	1,858,419	1,781,747	1.04
5,300	21,842	52,800	62,350	0.85
8,738	591,904	6,745,812	7,730,880	0.87
13,992	619,488	3,697,688	4,156,246	0.89

Table 2. A fill-in comparison of symmetric minimal degree and BBD orderings.

SECTION II:

AC ANALYSIS

KCL equations

$$-i_a + i_b + i_c = 0$$

$$-i_c + i_d + i_e = 0$$

Using node voltages

$$-I_{g1} + \frac{V_1}{R_1} + \frac{V_1 - V_2}{R_2} = 0$$

$$-\frac{V_1 - V_2}{R_2} + \frac{V_2}{R_3} + I_{g2} = 0$$

Grouping the terms

$$V_1\left(\frac{1}{R_2} + \frac{1}{R_2}\right) - \frac{1}{R_3}V_2 - I_{g1} = 0$$

$$-\frac{1}{R_2}V_1 + V_2\left(\frac{1}{R_2} + \frac{1}{R_3}\right) + I_{g2} = 0$$

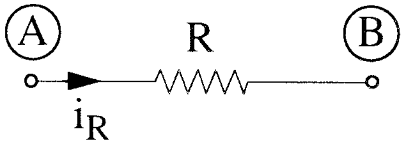
In matrix form

$$\begin{bmatrix} \left(\frac{1}{R_1}+\frac{1}{R_2}\right) & -\frac{1}{R_2} \\ -\frac{1}{R_2} & \left(\frac{1}{R_2}+\frac{1}{R_3}\right) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} - \begin{bmatrix} I_{g1} \\ -I_{g2} \end{bmatrix} = 0$$

GENERAL FORMAT: $G\,x - w = 0$

Stamps for circuit elements

CONTRIBUTION OF A RESISTOR



$$A: \quad \dots \; + i_R \; = \frac{V_A - V_B}{R}$$

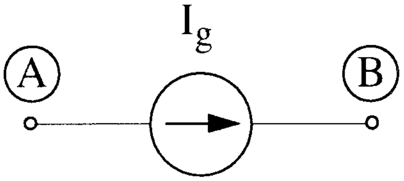
$$B: \quad \dots \; - i_R \; = -\frac{V_A + V_B}{R}$$

In nodal equations, the resistor appears in matrix G as

$$\begin{array}{cc} & \begin{array}{ccc} V_A & \cdots & V_B \end{array} \\ \begin{array}{c} A \\ \vdots \\ B \end{array} & \begin{bmatrix} \frac{1}{R} & & -\frac{1}{R} \\ & & \\ -\frac{1}{R} & & \frac{1}{R} \end{bmatrix} \begin{bmatrix} V_A \\ \vdots \\ V_B \end{bmatrix} \end{array}$$

This contribution is referred to as the *stamp* corresponding to resistor R.

CONTRIBUTION OF AN INDEPENDENT CURRENT SOURCE



$$A: \ \ldots \ +I_g$$

$$B: \ \ldots \ -I_g$$

In the nodal equations, the current source appears in vector w as

$$- \begin{array}{c} A \\ \vdots \\ B \end{array} \begin{bmatrix} -I_g \\ \vdots \\ I_g \end{bmatrix}$$

EXAMPLE 1 (redone using stamps)

Stamps for resistors (contribution to G only):

$$R_1: \quad \begin{array}{c} \\ 1 \\ 0 \end{array} \begin{array}{c} \begin{array}{cc} V_1 & V_0 \end{array} \\ \begin{bmatrix} \frac{1}{R_1} & -\frac{1}{R_1} \\ -\frac{1}{R_1} & \frac{1}{R_1} \end{bmatrix} \end{array} \quad \Rightarrow \quad \begin{array}{cc} & V_1 \\ 1 & \begin{bmatrix} \frac{1}{R_1} \end{bmatrix} \end{array}$$

$$R_2: \quad \begin{array}{c|cc} & V_1 & V_2 \\ \hline 1 & \frac{1}{R_2} & -\frac{1}{R_2} \\ 2 & -\frac{1}{R_2} & \frac{1}{R_2} \end{array}$$

$$R_3: \quad \begin{array}{c|cc} & V_2 & V_0 \\ \hline 2 & \frac{1}{R_3} & -\frac{1}{R_3} \\ 0 & -\frac{1}{R_3} & \frac{1}{R_3} \end{array} \quad \Rightarrow \quad \begin{array}{cc} & V_2 \\ 2 & \begin{bmatrix} \frac{1}{R_3} \end{bmatrix} \end{array}$$

Stamps for current sources (contribution to w only)

$$\begin{array}{c} 0 \\ 1 \end{array} \begin{bmatrix} -I_{g1} \\ I_{g1} \end{bmatrix} \Rightarrow 1 \begin{bmatrix} I_{g1} \end{bmatrix}$$

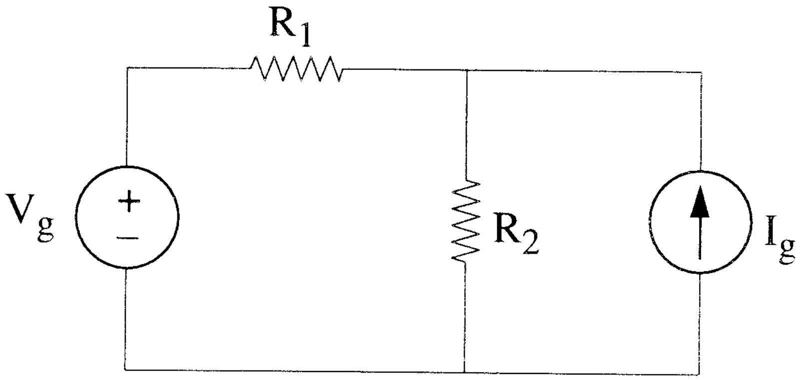
$$\begin{array}{c} 2 \\ 0 \end{array} \begin{bmatrix} -I_{g2} \\ I_{g2} \end{bmatrix} \Rightarrow 2 \begin{bmatrix} -I_{g2} \end{bmatrix}$$

Combining all the contributions

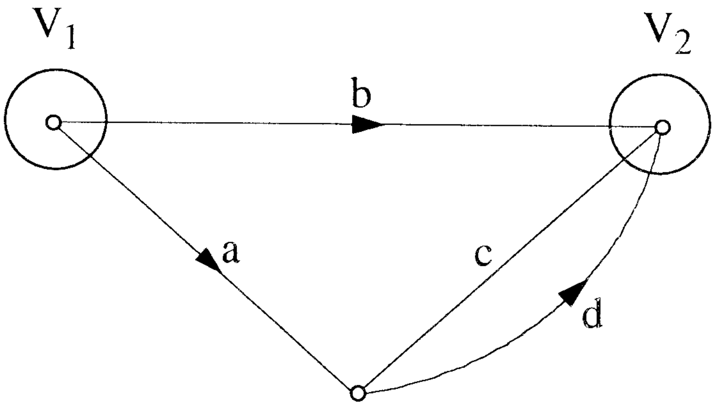
$$\begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} \qquad V_1 \qquad & \qquad V_2 \qquad \end{array} \\ \begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_2} & -\frac{1}{R_2} \\ -\frac{1}{R_2} & \frac{1}{R_2} + \frac{1}{R_3} \end{bmatrix} \end{array} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} - \begin{array}{c} 1 \\ 2 \end{array} \begin{bmatrix} I_{g1} \\ -I_{g2} \end{bmatrix} = 0$$

This is exactly what we had before.

EXAMPLE 2



Graph



KCL equations

$$i_a + i_b = 0$$

$$-i_b + i_c - i_d = 0$$

In this example we have a voltage source, and we *can not* express i_a in terms of node voltages. Therefore, we will need an extra equation (so-called *compensating equation*).

$$i_a + \frac{V_1 - V_2}{R_1} = 0$$

$$-\frac{V_1 - V_2}{R_1} + \frac{V_2}{R_2} - I_g = 0$$

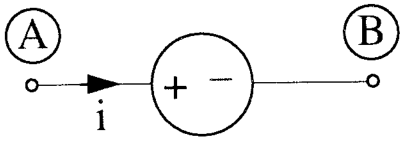
$$V_1 - V_g = 0$$

In matrix form

$$\begin{array}{c|ccc|}
 & V_1 & V_2 & i_a \\
\hline
1 & \frac{1}{R_1} & -\frac{1}{R_2} & 1 \\
2 & -\frac{1}{R_1} & \left(\frac{1}{R_1} + \frac{1}{R_2}\right) & 0 \\
COM & 1 & 0 & 0
\end{array}
\begin{bmatrix} V_1 \\ V_2 \\ i_a \end{bmatrix} - \begin{array}{c} 1 \\ 2 \\ COM \end{array} \begin{bmatrix} 0 \\ I_g \\ V_g \end{bmatrix} = 0$$

GENERAL FORMAT: $G\,x - w = 0$ is still valid, but x additionally contains current i_a.

CONTRIBUTION OF AN INDEPENDENT VOLTAGE SOURCE



$$A: \; \ldots \; +i$$

$$B: \; \ldots \; -i$$

$$COM: \; V_A - V_B - V_g = 0$$

In nodal equations, the voltage source appears *both* in G *and* in w

$$\begin{array}{cccc} & V_A & V_B & i \\ A & 0 & 0 & 1 \\ B & 0 & 0 & -1 \\ COM & 1 & -1 & 0 \end{array} \begin{bmatrix} V_A \\ V_B \\ i \end{bmatrix} - \begin{array}{cc} 1 & 0 \\ 2 & 0 \\ COM & V_g \end{array}$$

EXAMPLE 2 (redone using stamps)

Stamps for resistors (contribution to G only):

$$R_1: \quad \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} V_1 & V_2 \end{array} \\ \left[\begin{array}{cc} \frac{1}{R_1} & -\frac{1}{R_1} \\ -\frac{1}{R_1} & \frac{1}{R_1} \end{array} \right] \end{array}$$

$$R_2: \quad \begin{array}{c} \\ 2 \end{array} \begin{array}{c} V_2 \\ \left[\frac{1}{R_2} \right] \end{array}$$

Stamp for current source (contribution to w only):

$$I_g: \quad \begin{array}{c} 0 \\ 2 \end{array} \left[\begin{array}{c} -I_g \\ I_g \end{array} \right] \Rightarrow 2 \left[I_g \right]$$

Stamp for voltage source (contribution to both G and w)

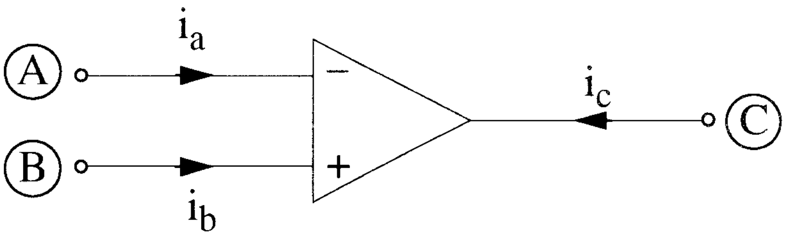
$$V_g: \quad \begin{array}{c|ccc|} & V_1 & V_0 & i \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ COM & 1 & -1 & 0 \end{array} \; ; \quad \begin{array}{c|c|} 1 & 0 \\ 0 & 0 \\ COM & V_g \end{array}$$

which becomes

$$V_g: \quad \begin{array}{c|cc|} & V_1 & i \\ 1 & 0 & 1 \\ COM & 1 & 0 \end{array} \; ; \quad \begin{array}{c|c|} 1 & 0 \\ COM & V_g \end{array}$$

Combining all the stamps we obtain exactly the same result as before. From this point on, we will write our equations using stamps only.

CONTRIBUTION OF AN OPERATIONAL AMPLIFIER



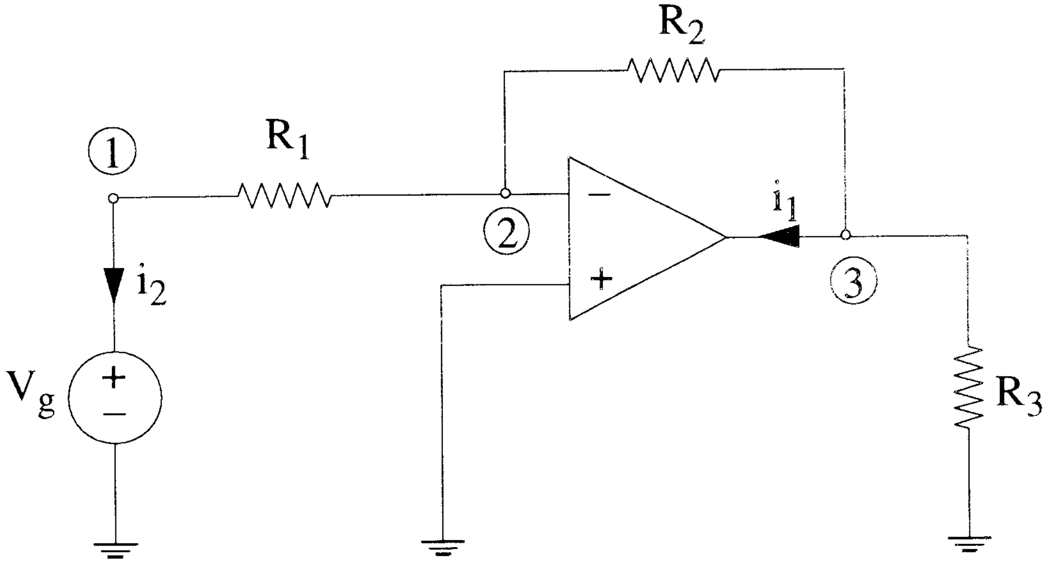
$A: \dots + i_a = 0$ $\qquad$ $C: \dots + i_c$

$B \dots + i_b = 0$ $\qquad$ $COM: \quad V_A - V_B = 0$

In nodal equations, the op-amp appears in G *only*, as

$$\begin{array}{c} \\ A \\ B \\ C \\ COM \end{array} \begin{array}{c} \begin{array}{cccc} V_A & V_B & V_C & i_c \end{array} \\ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ \left[\begin{array}{c} V_A \\ V_B \\ V_C \\ i \end{array}\right] \end{array}$$

EXAMPLE 3



Stamps for resistors (contribution to *G* only):

$$R_1: \quad \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} V_1 & V_0 \end{array} \\ \left[\begin{array}{cc} \frac{1}{R_1} & -\frac{1}{R_1} \\ -\frac{1}{R_1} & \frac{1}{R_1} \end{array}\right] \end{array}$$

$$R_2: \quad \begin{array}{c} \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} V_2 & V_3 \end{array} \\ \left[\begin{array}{cc} \frac{1}{R_2} & -\frac{1}{R_2} \\ -\frac{1}{R_2} & \frac{1}{R_2} \end{array}\right] \end{array} \quad ; \quad R_3: \quad 3 \begin{array}{c} V_3 \\ \left[\frac{1}{R_3} \right] \end{array}$$

Stamp for the op-amp (contribution to G only)

$$\begin{array}{c} \\ A \\ B \\ C \\ COM\ 1 \end{array} \begin{array}{c} \begin{array}{cccc} V_A & V_B & V_C & i_1 \end{array} \\ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \end{array} \right] \end{array} \Rightarrow \begin{array}{c} \\ 2 \\ 3 \\ COM\ 1 \end{array} \begin{array}{c} \begin{array}{ccc} V_2 & V_3 & i_1 \end{array} \\ \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right] \end{array}$$

Stamp for voltage source (contribution to both G and w)

$$\begin{array}{c} \\ A \\ B \\ COM\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} V_A & V_B & i_2 \end{array} \\ \left[\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{array} \right] \end{array} ; \quad \begin{array}{c} A \\ B \\ C_2 \end{array} \left[\begin{array}{c} 0 \\ 0 \\ V_g \end{array} \right]$$

This becomes

$$\begin{array}{c} \\ 1 \\ COM\ 2 \end{array} \begin{array}{cc} V_1 & i_2 \end{array} \hspace{-6em} \begin{array}{c} \\ \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right] \end{array} ; \quad \begin{array}{c} 1 \\ COM\ 2 \end{array} \left[\begin{array}{c} 0 \\ V_g \end{array}\right]$$

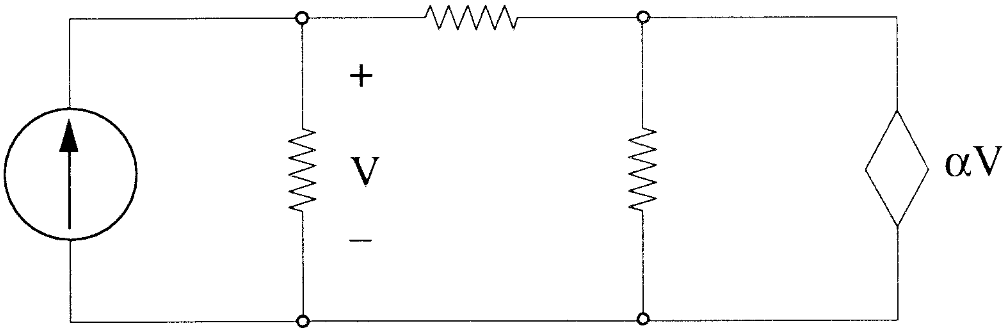
Combining all the stamps, we have

$$G = \begin{array}{c|ccccc} & V_1 & V_2 & V_3 & i_1 & i_2 \\ \hline 1 & \frac{1}{R_1} & -\frac{1}{R_2} & 0 & 0 & 1 \\ 2 & -\frac{1}{R_1} & \left(\frac{1}{R_1}+\frac{1}{R_2}\right) & -\frac{1}{R_2} & 0 & 0 \\ 3 & 0 & -\frac{1}{R_2} & \left(\frac{1}{R_2}+\frac{1}{R_3}\right) & 1 & 0 \\ COM\ 1 & 0 & 1 & 0 & 0 & 0 \\ COM\ 2 & 1 & 0 & 0 & 0 & 0 \end{array}$$

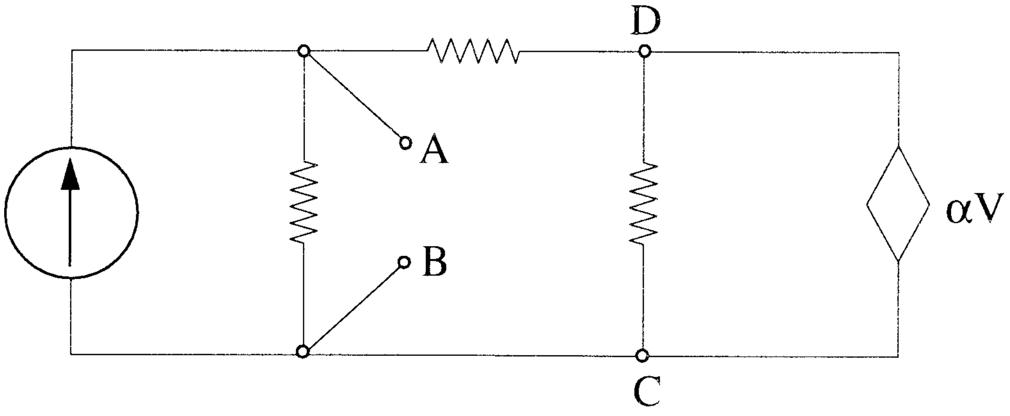
$$w = \begin{array}{r}1\\2\\3\\COM\ 1\\COM\ 2\end{array}\left[\begin{array}{c}0\\0\\0\\0\\V_g\end{array}\right]$$

CONTRIBUTION OF VOLTAGE-CONTROLLED SOURCES

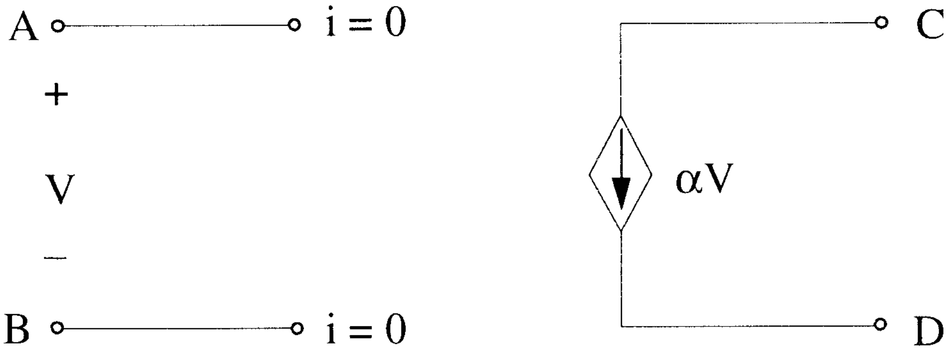
General format



It will be convenient to think of a voltage-controlled source as a *two-port*



Voltage-controlled current source



$A \ldots i = 0$

$B \ldots i = 0$

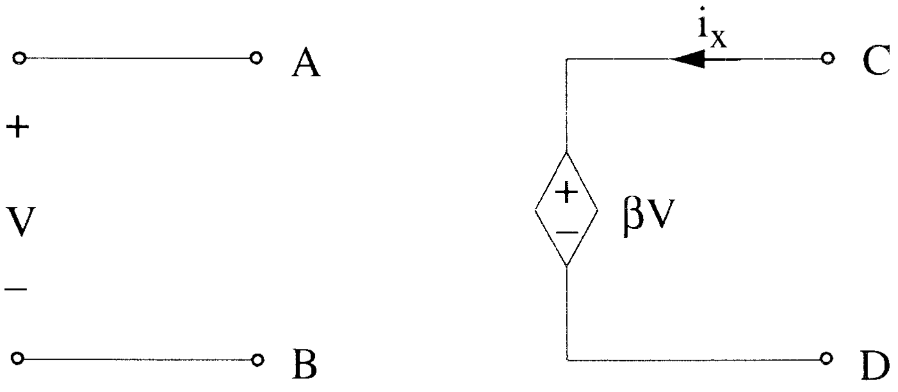
$C \ldots +\alpha(V_A - V_B)$

$D \ldots -\alpha(V_A - V_B)$

Stamp for voltage-controlled current source

$$\begin{array}{c|cccc} & V_A & V_B & V_C & V_D \\ \hline A & 0 & 0 & 0 & 0 \\ B & 0 & 0 & 0 & 0 \\ C & \alpha & -\alpha & 0 & 0 \\ D & -\alpha & \alpha & 0 & 0 \end{array}$$

Voltage-controlled voltage source



$A \ldots i = 0$ $\qquad$ $C \ldots i_x$

$B \ldots i = 0$ $\qquad$ $D \ldots -i_x$

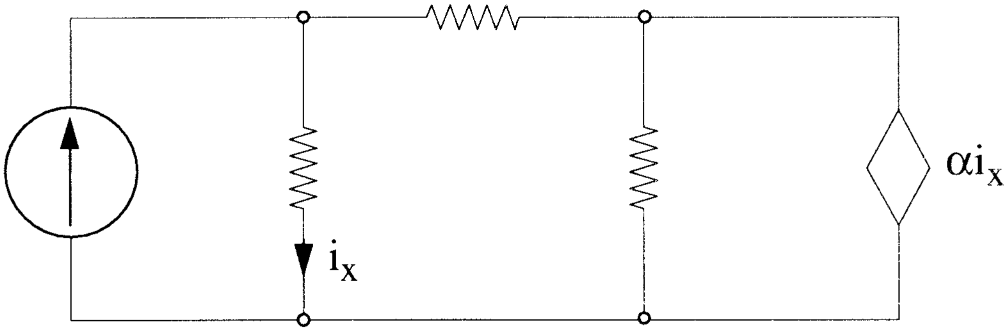
$$COM: V_C - V_D - \beta\,(V_A - V_B) = 0$$

Stamp for voltage-controlled voltage source

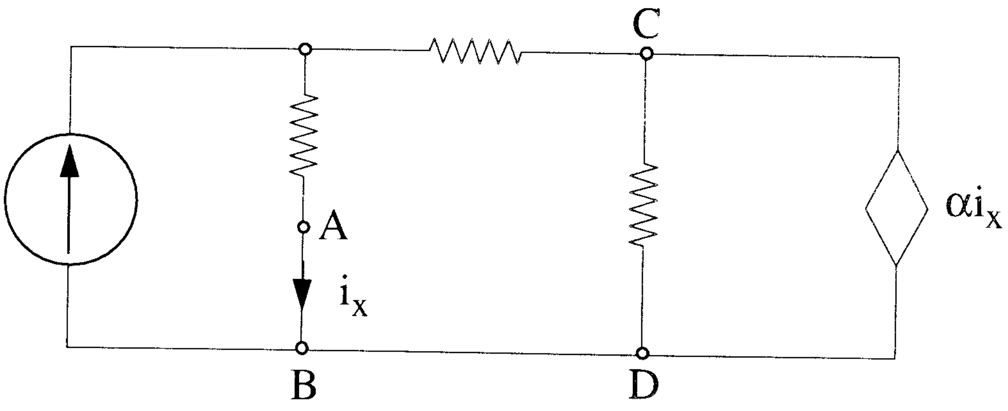
$$\begin{array}{c|ccccc|}
 & V_A & V_B & V_C & V_D & i_x \\
A & 0 & 0 & 0 & 0 & 0 \\
B & 0 & 0 & 0 & 0 & 0 \\
C & 0 & 0 & 0 & 0 & 1 \\
D & 0 & 0 & 0 & 0 & -1 \\
COM & -\beta & \beta & 1 & -1 & 0
\end{array}$$

CONTRIBUTION OF CURRENT-CONTROLLED SOURCES

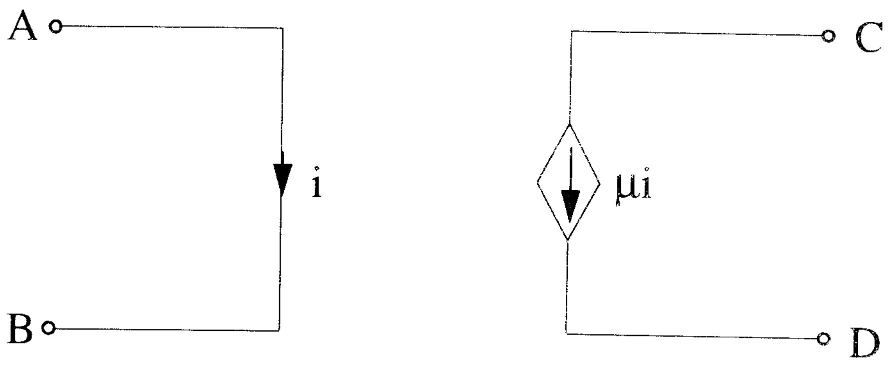
General format



It will be convenient to think of a current-controlled source as a *two-port*



Current-controlled current source



$A \ldots +i$ $\qquad$ $C \ldots \mu i$

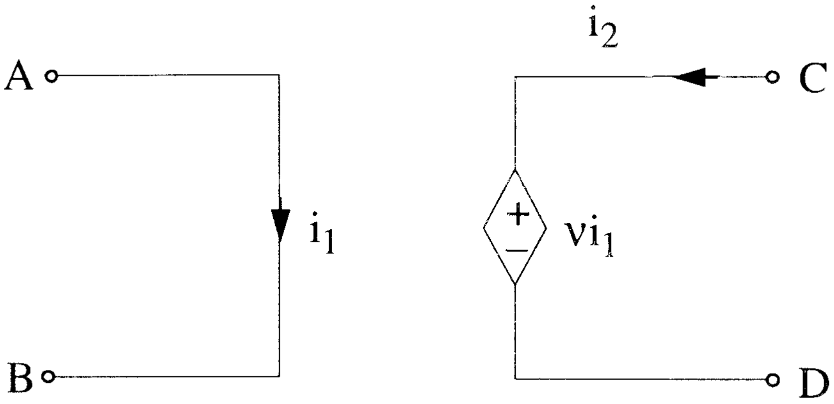
$B \ldots -i$ $\qquad$ $D \ldots -\mu i$

$COM: \; V_A - V_B = 0$

Stamp for current-controlled current source

$$\begin{array}{c|ccccc|}
 & V_A & V_B & V_C & V_D & i_x \\
A & 0 & 0 & 0 & 0 & 1 \\
B & 0 & 0 & 0 & 0 & -1 \\
C & 0 & 0 & 0 & 0 & \mu \\
D & 0 & 0 & 0 & 0 & -\mu \\
COM & 1 & -1 & 0 & 0 & 0
\end{array}$$

Current-controlled voltage source



$A \ldots +i_1$ $C \ldots i_2$

$B \ldots -i_1$ $D \ldots -i_2$

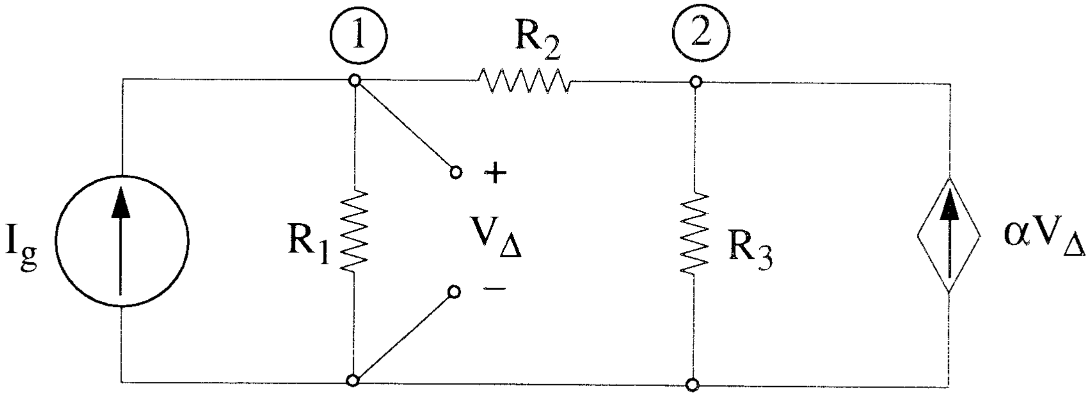
$COM\,1: \; V_A - V_B = 0$

$COM\,2: \; V_C - V_D - \nu\, i_1 = 0$

Stamp for current-controlled voltage source

$$
\begin{array}{c|cccccc|}
 & V_A & V_B & V_C & V_D & i_1 & i_2 \\
\hline
A & 0 & 0 & 0 & 0 & 1 & 0 \\
B & 0 & 0 & 0 & 0 & -1 & 0 \\
C & 0 & 0 & 0 & 0 & 0 & 1 \\
D & 0 & 0 & 0 & 0 & 0 & -1 \\
COM\ 1 & 1 & -1 & 0 & 0 & 0 & 0 \\
COM\ 2 & 0 & 0 & 1 & -1 & -\nu & 0 \\
\hline
\end{array}
$$

EXAMPLE 4



Stamps for resistors (contribution to G only):

$$R_1: \quad \begin{array}{c c} & V_1 \\ 1 & \left[\dfrac{1}{R_1} \right] \end{array}$$

$$R_2: \quad \begin{array}{c c c} & V_1 & V_2 \\ 1 & \dfrac{1}{R_2} & -\dfrac{1}{R_2} \\ 2 & -\dfrac{1}{R_2} & \dfrac{1}{R_2} \end{array} \quad ; \quad R_3: \quad \begin{array}{c c} & V_2 \\ 2 & \left[\dfrac{1}{R_3} \right] \end{array}$$

Stamp for current source (contribution to w only)

$$\begin{array}{c c} 0 & -I_g \\ 1 & I_g \end{array} \Rightarrow \begin{array}{c c} 1 & \left[I_g \right] \end{array}$$

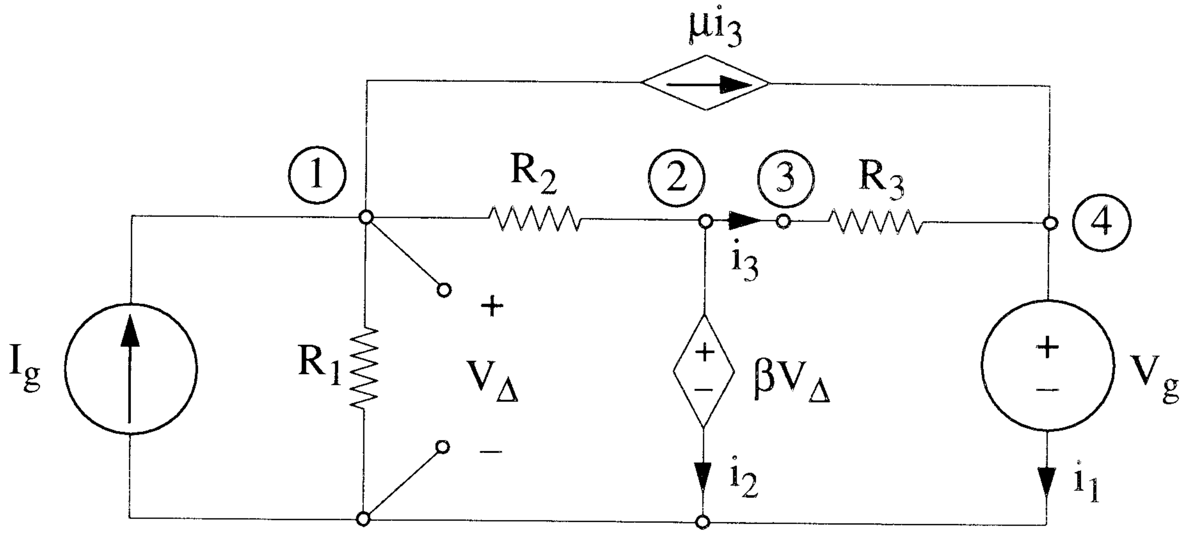
Stamp for voltage-controlled current source (contribution to *G* only)

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} V_A & V_B & V_C & V_D \end{array} \\ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & -\alpha & 0 & 0 \\ -\alpha & \alpha & 0 & 0 \end{array}\right] \end{array} \Rightarrow \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} V_1 & V_2 \end{array} \\ \left[\begin{array}{cc} 0 & 0 \\ -\alpha & 0 \end{array}\right] \end{array}$$

Combining all the stamp contributions

$$\begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} V_1 & V_2 \end{array} \\ \left[\begin{array}{cc} \left(\frac{1}{R_1}+\frac{1}{R_2}\right) & -\frac{1}{R_2} \\ \left(-\frac{1}{R_2}-\alpha\right) & \left(\frac{1}{R_2}+\frac{1}{R_3}\right) \end{array}\right] \end{array} \left[\begin{array}{c} V_1 \\ V_2 \end{array}\right] - \begin{array}{c} 1 \\ 2 \end{array} \left[\begin{array}{c} I_g \\ 0 \end{array}\right] = 0$$

EXAMPLE 5



Stamps for resistors (contribution to G only):

$$R_1: \quad \begin{array}{cc} & V_1 \\ 1 & \left[\dfrac{1}{R_1} \right] \end{array}$$

$$R_2: \quad \begin{array}{c|cc} & V_1 & V_2 \\ \hline 1 & \dfrac{1}{R_2} & -\dfrac{1}{R_2} \\ 2 & -\dfrac{1}{R_2} & \dfrac{1}{R_2} \end{array} \quad ; \quad R_3: \quad \begin{array}{c|cc} & V_3 & V_4 \\ \hline 3 & \dfrac{1}{R_3} & -\dfrac{1}{R_3} \\ 4 & -\dfrac{1}{R_3} & \dfrac{1}{R_3} \end{array}$$

Stamp for current source (contribution to w only)

$$\begin{array}{c} 0 \\ 1 \end{array} \begin{bmatrix} -I_g \\ I_g \end{bmatrix} \Rightarrow 1 \begin{bmatrix} I_g \end{bmatrix}$$

Stamp for voltage source (contribution to G and w)

$$\begin{array}{cc} & \begin{array}{ccc} V_A & V_B & i_2 \end{array} \\ \begin{array}{c} A \\ B \\ COM\ 1 \end{array} & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \end{array} ; \begin{bmatrix} 0 \\ 0 \\ V_g \end{bmatrix} \Rightarrow$$

$$\begin{array}{cc} & \begin{array}{cc} V_4 & i_1 \end{array} \\ \begin{array}{c} 4 \\ COM\ 1 \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array} ; \quad \begin{array}{cc} & W \\ \begin{array}{c} 4 \\ COM\ 1 \end{array} & \begin{bmatrix} 0 \\ V_g \end{bmatrix} \end{array}$$

Stamp for voltage-controlled voltage source (contribution to G only)

$$\begin{array}{c|ccccc|} & V_A & V_B & V_C & V_D & i_2 \\ \hline A & 0 & 0 & 0 & 0 & 0 \\ B & 0 & 0 & 0 & 0 & 0 \\ C & 0 & 0 & 0 & 0 & 1 \\ D & 0 & 0 & 0 & 0 & -1 \\ COM\ 2 & -\beta & \beta & 1 & -1 & 0 \end{array} \Rightarrow \begin{array}{c|ccc|} & V_A & V_B & i_2 \\ \hline 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ COM\ 2 & -\beta & 1 & 0 \end{array}$$

Stamp for current-controlled current source (contribution to G only)

$$\begin{array}{c|ccccc|} & V_2 & V_3 & V_1 & V_4 & i_3 \\ \hline 2 & 0 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & \mu \\ 4 & 0 & 0 & 0 & 0 & -\mu \\ COM\ 3 & 1 & -1 & 0 & 0 & 0 \end{array}$$

Combining all the stamps, matrix G will be:

$$\begin{array}{c|ccccccc|}
 & V_1 & V_2 & V_3 & V_4 & i_1 & i_2 & i_3 \\
\hline
1 & \left(\frac{1}{R_1}+\frac{1}{R_2}\right) & -\frac{1}{R_2} & 0 & 0 & 0 & 0 & \mu \\
2 & -\frac{1}{R_2} & \frac{1}{R_2} & 0 & 0 & 0 & 1 & 1 \\
3 & 0 & 0 & \frac{1}{R_3} & -\frac{1}{R_3} & 0 & 0 & -1 \\
4 & 0 & 0 & -\frac{1}{R_3} & \frac{1}{R_3} & 1 & 0 & -\mu \\
COM\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
COM\ 2 & -\beta & 1 & 0 & 0 & 0 & 0 & 0 \\
COM\ 3 & 0 & 1 & -1 & 0 & 0 & 0 & 0
\end{array}$$

and vector *w* will be

$$\begin{bmatrix} I_g \\ 0 \\ 0 \\ 0 \\ V_g \\ 0 \\ 0 \end{bmatrix}$$

AC ANALYSIS OF LINEAR CIRCUITS

In AC analysis we use phasors, so inductors and capacitors can be easily incorporated. The stamps for these elements are provided below.

Capacitors

$$\begin{aligned} A \ \ldots \ +I &= j\omega C(V_A - V_B) \\ B \ \ldots \ -I &= -j\omega C(V_A - V_B) \end{aligned} \quad \Rightarrow \quad \begin{array}{cc} & \begin{array}{ccc} V_A & \cdots & V_B \end{array} \\ \begin{array}{c} A \\ \vdots \\ B \end{array} & \begin{bmatrix} j\omega C & & -j\omega C \\ & & \\ -j\omega C & & j\omega C \end{bmatrix} \end{array}$$

Inductors

$$\begin{aligned} A \ \ldots \ +I &= \frac{1}{j\omega L}(V_A - V_B) \\ B \ \ldots \ -I &= -\frac{1}{j\omega L}(V_A - V_B) \end{aligned} \quad \Rightarrow \quad \begin{array}{cc} & \begin{array}{ccc} V_A & \cdots & V_B \end{array} \\ \begin{array}{c} A \\ \\ B \end{array} & \begin{bmatrix} \frac{1}{j\omega L} & & -\frac{1}{j\omega L} \\ & & \\ -\frac{1}{j\omega L} & & \frac{1}{j\omega L} \end{bmatrix} \end{array}$$

The general equation format now becomes

$$\left[G + j\omega C + \frac{1}{j\omega} L \right] x = w$$

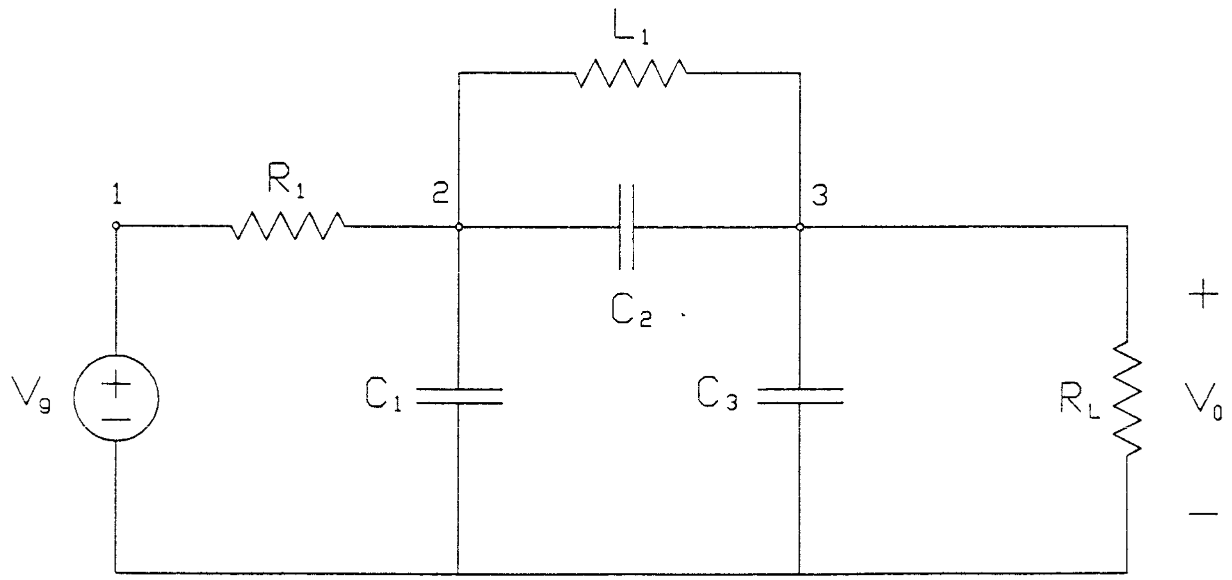
where C and L are matrices corresponing to capacitors and inductors. Basically, this is the same format as before, only the matrix elements are complex numbers.

Application to Filters

An important application of AC analysis is in the design of filters.

EXAMPLE

The circuit below represents a *passive low pass filter.*



Stamps for resistors (contribution to G only):

$$R_1 : \quad \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{bmatrix} \dfrac{1}{R_1} & -\dfrac{1}{R_1} \\ -\dfrac{1}{R_1} & \dfrac{1}{R_1} \end{bmatrix} \end{array}$$

$$R_L : \quad \begin{array}{cc} & x_3 \\ 3 & \begin{bmatrix} \dfrac{1}{R_L} \end{bmatrix} \end{array}$$

Stamp for voltage source (contribution to both G and w)

$$V_g : \quad \begin{array}{cc} & \begin{array}{cc} x_1 & x_4 \end{array} \\ \begin{array}{c} 1 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array} \; ; \quad \begin{array}{c} 1 \\ 4 \end{array} \begin{bmatrix} 0 \\ V_g \end{bmatrix}$$

Stamps for capacitors (contribution to C only):

$$C_1: \quad \begin{array}{c|c} & x_2 \\ \hline 2 & C_1 \end{array}$$

$$C_2: \quad \begin{array}{c|cc} & x_2 & x_3 \\ \hline 2 & C_2 & -C_2 \\ 3 & -C_2 & C_2 \end{array}$$

$$C_3: \quad \begin{array}{c|c} & x_3 \\ \hline 3 & C_3 \end{array}$$

Stamp for inductor (contribution to L only):

$$L_1: \quad \begin{array}{c|cc} & x_2 & x_3 \\ \hline 2 & L_1 & -L_1 \\ 3 & -L_1 & L_1 \end{array}$$

Combining all the stamp contributions

$$G = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc} \frac{1}{R_1} & -\frac{1}{R_1} & 0 & 1 \\ -\frac{1}{R_1} & \frac{1}{R_1} & 0 & 0 \\ 0 & 0 & \frac{1}{R_L} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \quad ; \quad w = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$L = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & L_1 & -L_1 & 0 \\ 0 & -L_1 & L_1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$C = \left[\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ 0 & 0 & 0 & 0 \\ 0 & (C_1 + C_2) & -C_2 & 0 \\ 0 & -C_2 & (C_2 + C_3) & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

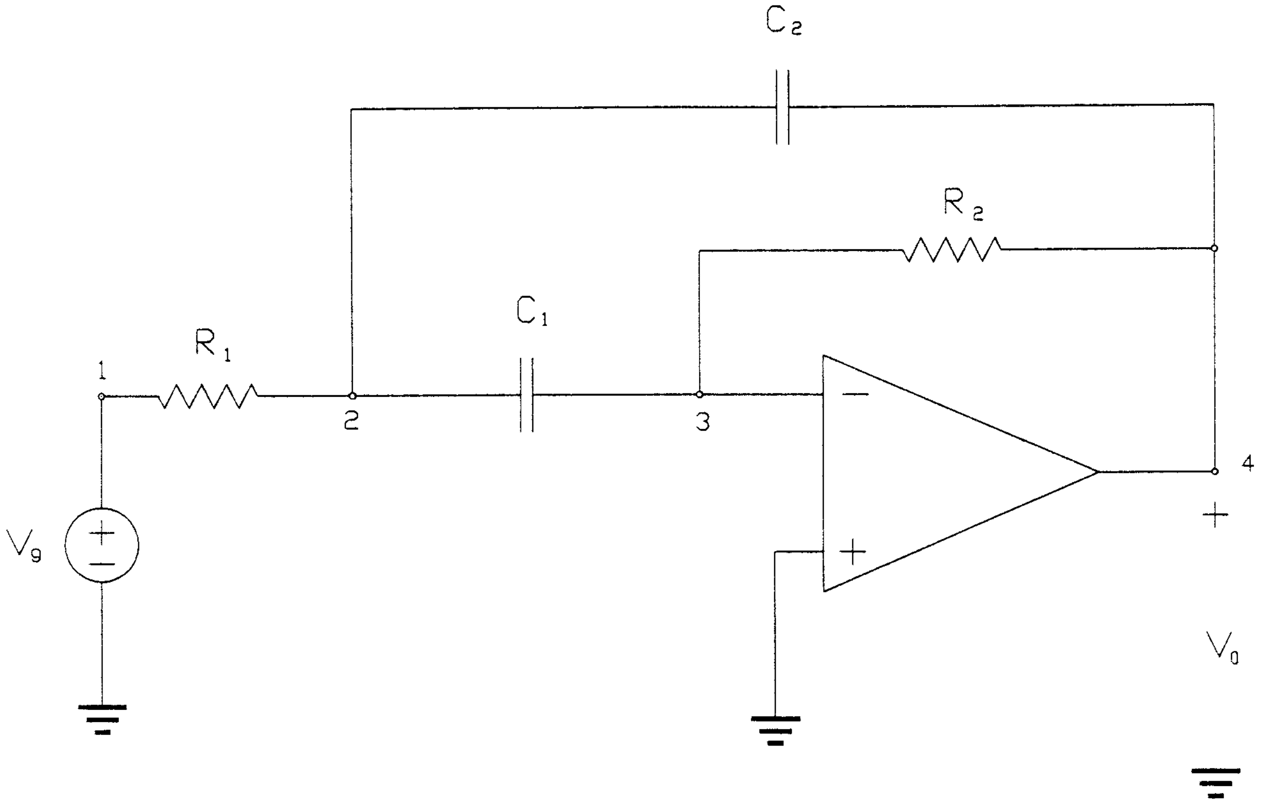
By solving equation

$$\left[G + j\omega C + \frac{1}{j\omega} L \right] x = w$$

for different values of ω, we can obtain the *frequency response* of this circuit (both magnitude and phase, since x is complex).

EXAMPLE

The circuit below represents a *active band pass filter.*



Stamps for resistors (contribution to G only):

$$R_1: \quad \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{bmatrix} \frac{1}{R_1} & -\frac{1}{R_1} \\ -\frac{1}{R_1} & \frac{1}{R_1} \end{bmatrix} \end{array}$$

$$R_2: \quad \begin{array}{c} \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cc} x_3 & x_4 \end{array} \\ \begin{bmatrix} \frac{1}{R_2} & -\frac{1}{R_2} \\ -\frac{1}{R_2} & \frac{1}{R_2} \end{bmatrix} \end{array}$$

Stamp for voltage source (contribution to both G and w)

$$V_g: \quad \begin{array}{c} \\ 1 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} x_1 & x_5 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array} ; \quad \begin{array}{c} 1 \\ 5 \end{array} \begin{bmatrix} 0 \\ V_g \end{bmatrix}$$

Stamps for capacitors (contribution to C only):

$$C_1: \quad \begin{array}{c} \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} x_2 & x_3 \end{array} \\ \begin{bmatrix} C_1 & -C_1 \\ -C_1 & C_1 \end{bmatrix} \end{array}$$

$$C_2: \quad \begin{array}{c} \\ 2 \\ 4 \end{array} \begin{array}{c} \begin{array}{cc} x_2 & x_4 \end{array} \\ \begin{bmatrix} C_2 & -C_2 \\ -C_2 & C_2 \end{bmatrix} \end{array}$$

Stamp for the op-amp (contribution to G only)

$$OA: \quad \begin{array}{c} \\ 3 \\ 4 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccc} x_3 & x_4 & x_6 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

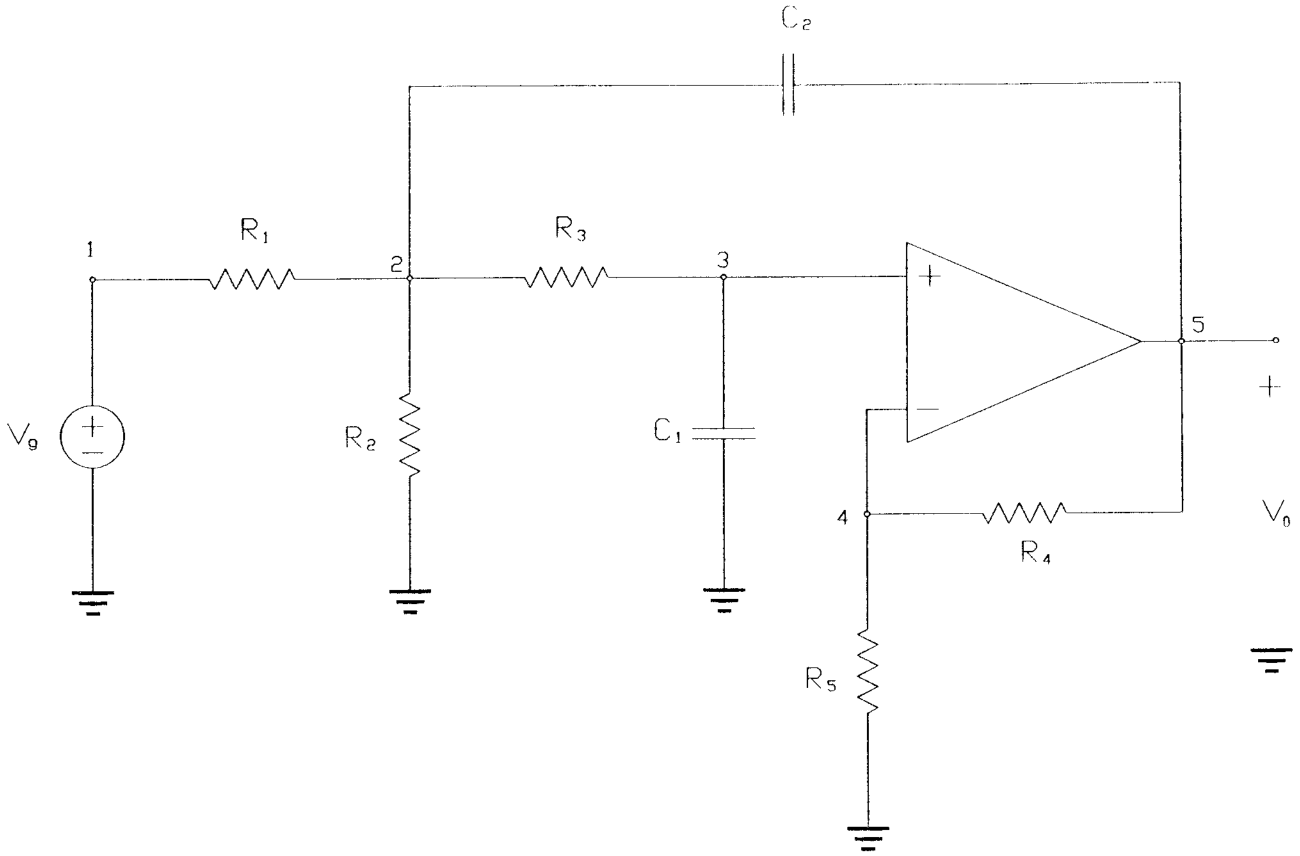
In this case $L = 0$, so the equations have the form

$$\left[G + j\omega C \right] x = w$$

Again, we can solve it for different values of ω and obtain the frequency response.

EXAMPLE

This circuit is an *active low pass filter.*



Stamps for resistors (contribution to G only):

$$R_1: \begin{array}{c|cc} & x_1 & x_2 \\ \hline 1 & \frac{1}{R_1} & -\frac{1}{R_1} \\ 2 & -\frac{1}{R_1} & \frac{1}{R_1} \end{array} \quad ; \quad R_2: \begin{array}{c|c} & x_2 \\ \hline 2 & \frac{1}{R_2} \end{array}$$

$$R_3: \begin{array}{c|cc} & x_2 & x_3 \\ \hline 2 & \frac{1}{R_3} & -\frac{1}{R_3} \\ 3 & -\frac{1}{R_3} & \frac{1}{R_3} \end{array}$$

$$R_4: \begin{array}{c|cc} & x_4 & x_5 \\ \hline 4 & \frac{1}{R_4} & -\frac{1}{R_4} \\ 5 & -\frac{1}{R_4} & \frac{1}{R_4} \end{array} \quad ; \quad R_5: \begin{array}{c|c} & x_4 \\ \hline 4 & \frac{1}{R_5} \end{array}$$

Stamp for voltage source (contribution to both G and w)

$$V_g : \quad \begin{array}{c|cc|} & x_1 & x_6 \\ 1 & 0 & 1 \\ 6 & 1 & 0 \end{array} \; ; \quad \begin{array}{c|c|} 1 & 0 \\ 6 & V_g \end{array}$$

Stamps for capacitors (contribution to C only):

$$C_1 : \quad \begin{array}{c|c|} & x_3 \\ 3 & C_1 \end{array}$$

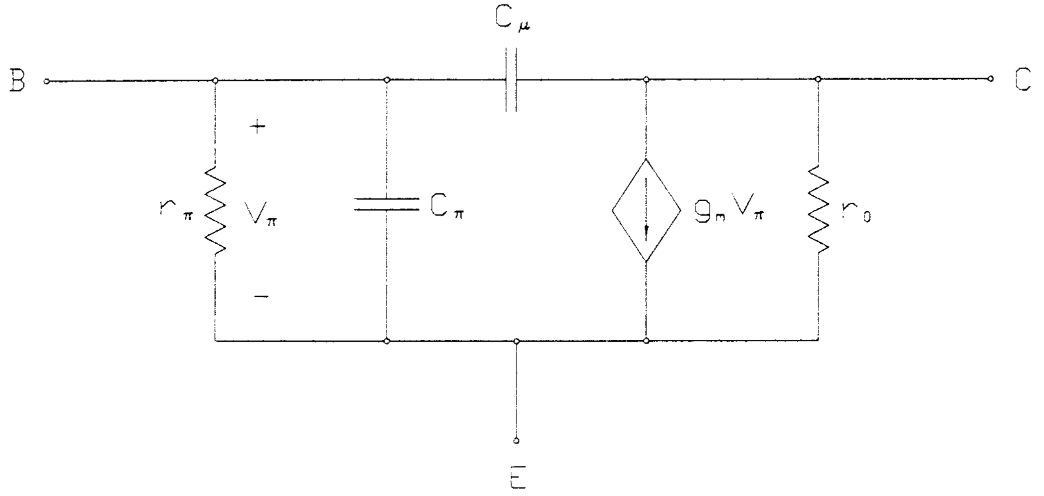
$$C_2 : \quad \begin{array}{c|cc|} & x_2 & x_5 \\ 2 & C_2 & -C_2 \\ 5 & -C_2 & C_2 \end{array}$$

Stamp for op amp (contribution to G only)

$$OA: \begin{array}{c} \\ 4 \\ 3 \\ 5 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccc} x_4 & x_3 & x_5 & x_7 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix} \end{array}$$

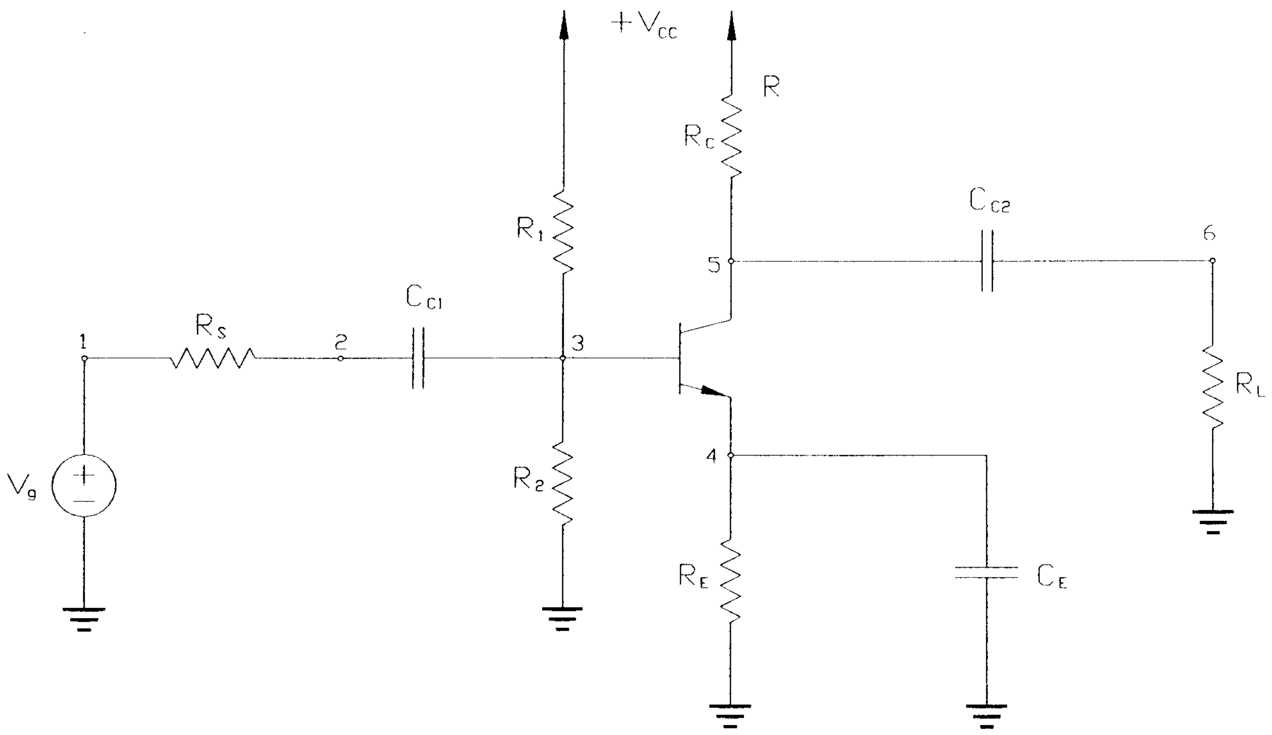
Application to Linear Amplifiers

AC analysis can be used to evaluate the frequency response of linear amplifiers. We illustrate this process by considering the *common emitter amplifier*; in doing so, we will use the following small signal transistor model



EXAMPLE

The common emitter amplifier circuit is shown below.



In this circuit, the element values are:

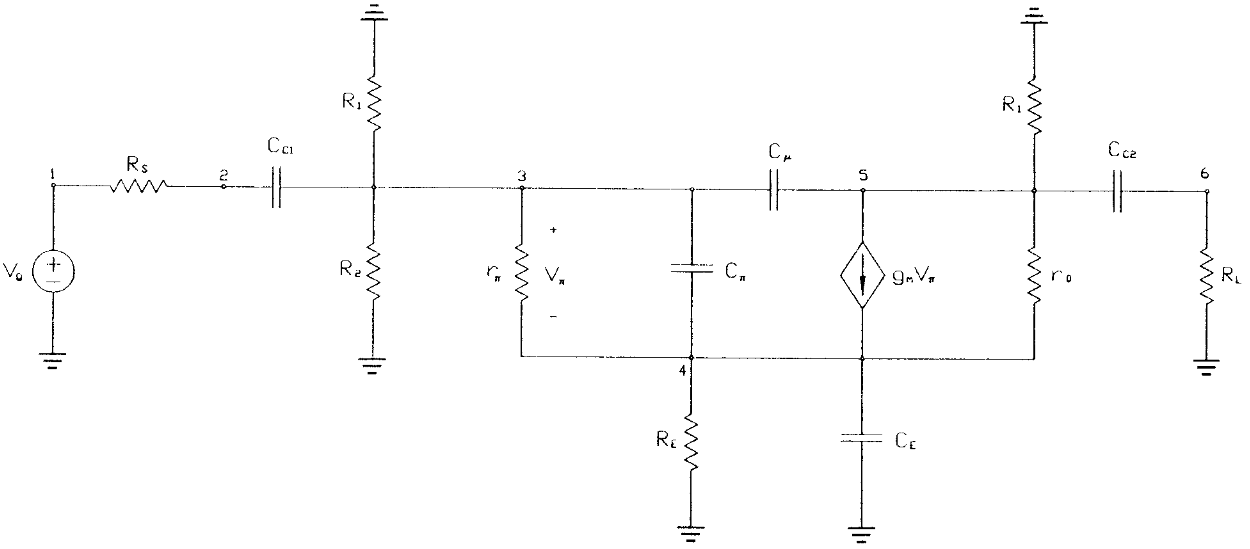
R_1 = 8K; R_2 = 4K; R_C = 6K; R_E = 3.3K; R_L = 8K; $C_{C1} = C_{C2}$ = 1μF;

C_E = 10μF; V_{CC} = 12V.

The parameters of the small signal model can be obtained from a DC analysis as:

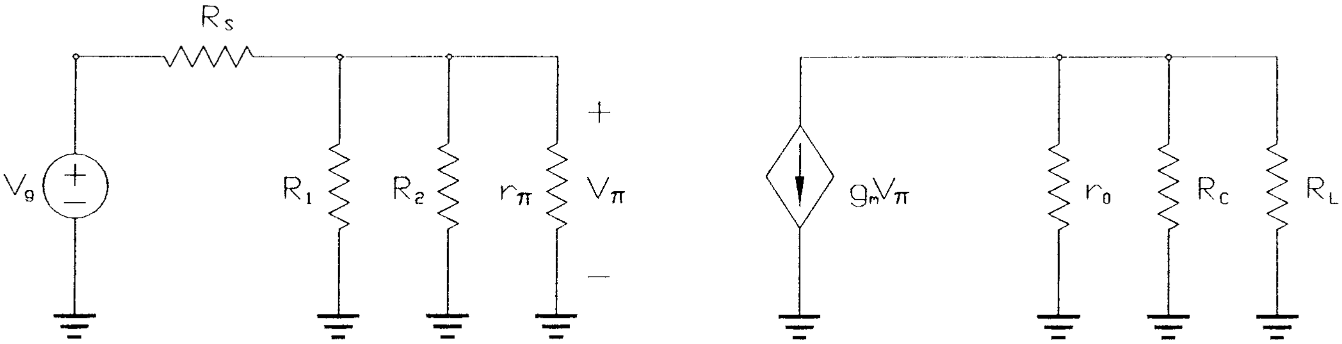
g_m = 36 mA/V; r_π = 2.6K; r_0 = 103K; C_π = 17pF; C_μ = 2.5pF.

The linearized circuit that we will use for an AC analysis is shown below.



COMMENT 1: At low frequencies the gain will be affected by coupling capacitors C_{C1}, C_{C2} and C_E, while the high frequency behavior is determined by the internal capacitances of the transistor (C_π and C_μ).

COMMENT 2: For intermediate frequencies (*i. e.* in the kHz range), coupling capacitors can be approximated by *short circuits*, and the internal capacitances can be disregarded. The resulting simplified model becomes



From this model we can easily determine the gain as

$$V_0 = -g_m \cdot R_B \cdot \frac{R_A}{R_A + R_S} \cdot V_g$$

where $R_A = R_1 \parallel R_2 \parallel r_\pi$ and $R_B = R_C \parallel R_L \parallel r_0$. Obviously, in this range the gain is *independent* of the frequency, and the characteristic is "flat".

To perform a complete AC simulation, we need to consider *all* the capacitors. The individual element stamps are shown below.

Stamps for resistors (contribution to G only):

$$R_S: \quad \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{bmatrix} \frac{1}{R_S} & -\frac{1}{R_S} \\ -\frac{1}{R_S} & \frac{1}{R_S} \end{bmatrix} \end{array} \quad ; \quad R_1: \quad 3 \begin{array}{c} x_3 \\ \begin{bmatrix} \frac{1}{R_1} \end{bmatrix} \end{array}$$

$$r_\pi: \quad \begin{array}{c} \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cc} x_3 & x_4 \end{array} \\ \begin{bmatrix} \frac{1}{r_\pi} & -\frac{1}{r_\pi} \\ -\frac{1}{r_\pi} & \frac{1}{r_\pi} \end{bmatrix} \end{array} \quad ; \quad R_2: \quad 3 \begin{array}{c} x_3 \\ \begin{bmatrix} \frac{1}{R_2} \end{bmatrix} \end{array}$$

$$r_0: \quad \begin{array}{c|cc} & x_4 & x_5 \\ \hline 4 & \frac{1}{r_0} & -\frac{1}{r_0} \\ 5 & -\frac{1}{r_0} & \frac{1}{r_0} \end{array} \quad ; \qquad R_E: \quad \begin{array}{c|c} & x_4 \\ \hline 4 & \frac{1}{R_E} \end{array}$$

$$R_C: \quad \begin{array}{c|c} & x_5 \\ \hline 5 & \frac{1}{R_C} \end{array}$$

$$R_L: \quad \begin{array}{c|c} & x_6 \\ \hline 6 & \frac{1}{R_L} \end{array}$$

Stamp for controlled source (contribution to *G* only):

$$g_m: \quad \begin{array}{c|cc} & x_4 & x_5 \\ \hline 4 & -g_m & g_m \\ 5 & g_m & -g_m \end{array}$$

Stamp for voltage source (contribution to both G and w)

$$V_g: \quad \begin{array}{c|cc|} & x_1 & x_7 \\ \hline 1 & 0 & 1 \\ 7 & 1 & 0 \end{array} \quad ; \quad \begin{array}{c|c|} 1 & 0 \\ 7 & V_g \end{array}$$

Stamps for capacitors (contribution to C only):

$$C_\pi: \quad \begin{array}{c|cc|} & x_3 & x_4 \\ \hline 3 & C_\pi & -C_\pi \\ 4 & -C_\pi & C_\pi \end{array} \quad ; \quad C_\mu: \quad \begin{array}{c|cc|} & x_3 & x_5 \\ \hline 3 & C_\mu & -C_\mu \\ 5 & -C_\mu & C_\mu \end{array}$$

$$C_{C1}: \quad \begin{array}{c|cc|} & x_2 & x_3 \\ \hline 2 & C_{C1} & -C_{C1} \\ 3 & -C_{C1} & C_{C1} \end{array} \quad ; \quad C_E: \quad \begin{array}{c|c|} & x_4 \\ \hline 4 & C_E \end{array}$$

$$C_{\mathrm{C2}}: \quad \begin{array}{c} \\ 5 \\ \\ 6 \end{array} \begin{array}{cc} x_5 & x_6 \\ \left[\begin{array}{cc} C_{\mathrm{C2}} & -C_{\mathrm{C2}} \\ & \\ -C_{\mathrm{C2}} & C_{\mathrm{C2}} \end{array} \right] \end{array}$$

SECTION III:

DC ANALYSIS

NONLINEAR ALGEBRAIC EQUATIONS

The simplest example of a nonlinear equation is a quadratic equation, such as

$$f(x) \equiv x^2 + 4x + 3 = 0$$

One feature that we can immediately observe is that nonlinear equations can have *multiple solutions*, even if there is only one variable. In contrast, systems of linear equations will have a *unique* solution whenever the matrix is non-singular.

Newton's Method

We will first consider nonlinear equations in *one* variable. The general form of these equations is

$$f(x) = 0$$

and they can be solved by *Newton's iterative method*

$$x(k+1) = x(k) - \left[f'(x(k))\right]^{-1} f(x(k))$$

You begin from some initial guess $x(0)$, and then generate a sequence of points using the formula above. Specifically,

$$x(1) = x(0) - \left[f'(x(0))\right]^{-1} f(x(0))$$

$$x(2) = x(1) - \left[f'(x(1))\right]^{-1} f(x(1))$$

.

.

.

and so on, until the sequence converges to a solution.

EXAMPLE

For the quadratic equation that was considered on the previous page, Newton's method will be

$$x(k+1) = x(k) - \left[2x(k) + 4\right]^{-1} \left[x(k)^2 + 4x(k) + 3\right]$$

The solution that we obtain will depend on the initial guess.

Case 1

If we choose $x(0) = 0$, we obtain the following sequence:

$x(1) = - 0.75$

$x(2) = - 0.975$

$x(3) = - 0.99695$

$x(4) = - 0.99999$

Obviously, from this initial condition the method converges to $x^* = -1$ after 4 iterations.

Case 2

If we choose $x(0) = -5$, we obtain a different sequence:

$x(1) = - 3.67$

$x(2) = - 3.133$

$x(3) = - 3.00784$

$x(4) = - 3.00003$

$x(5) = - 3$

In this case, Newton's method converges to $x^* = -3$ after 5 iterations.

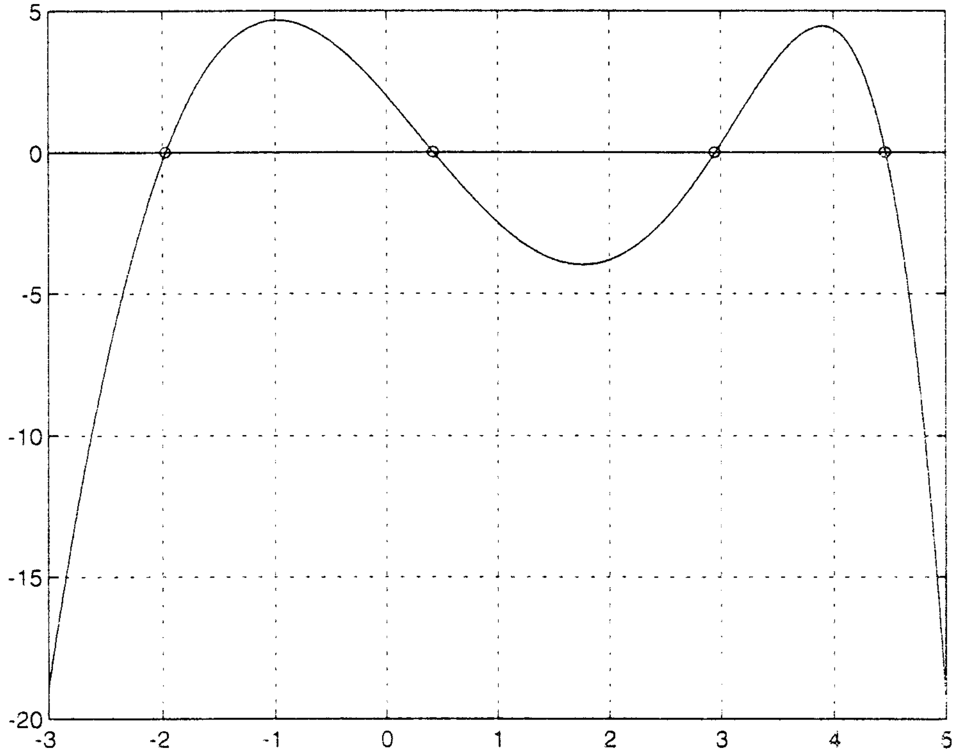
COMMENT. We can conclude from this example that the solution process depends heavily on the choice of $x(0)$. When the equation has multiple solutions, different initial approximations will lead to very different solutions.

EXAMPLE

This example is intended to further illustrate the importance of the inital approximation. Let us consider the following nonlinear equation

$$f(x) \equiv x^3 - x^2 - 6x - 2 - e^x + 5e^{\frac{x}{2}} = 0$$

A plot of this function is shown below, indicating that the equation has *four* different solutions.



In this case we have

$$f(x(k)) = x(k)^3 - x(k)^2 - 6x(k) - 2 - e^{x(k)} + 5e^{\frac{x(k)}{2}}$$

and

$$f'(x(k)) = 3x(k)^2 - 2x(k) - 6 - e^{x(k)} + \frac{5}{2}e^{\frac{x(k)}{2}}$$

Case 1

For $x(0) = 0$, Newton's method converges to $x^* = 0.4250802$ after 3 iterations.

Case 2

For $x(0) = 2$, Newton's method converges to $x^* = 4.4641297$ after 6 iterations.

Case 3

For $x(0) = -1$, Newton's method converges to $x^* = -1.9720659$ after 13 iterations.

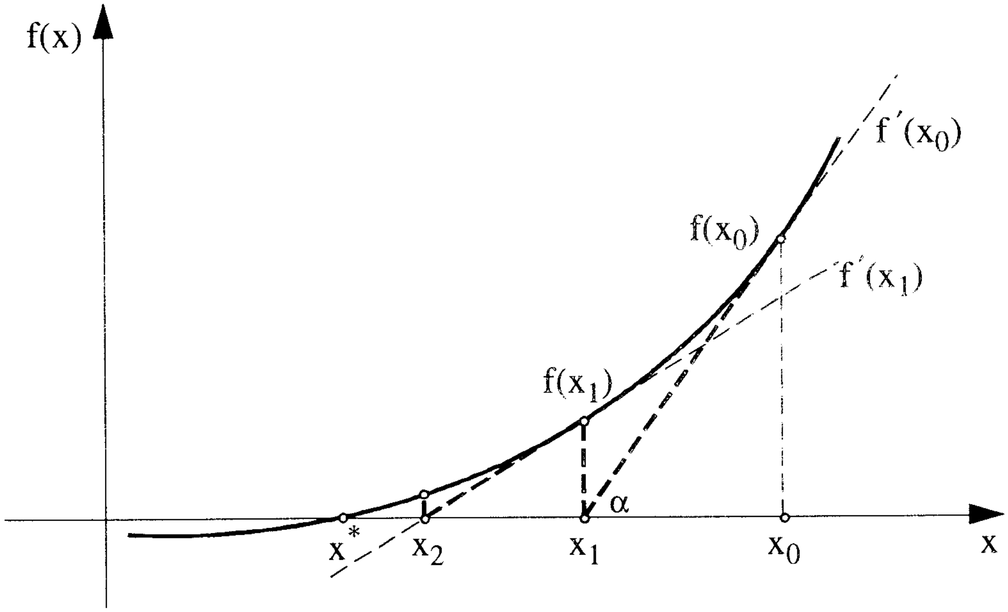
Case 4

For $x(0) = 3$, Newton's method converges to $x^* = 2.94683037$ after 3 iterations.

COMMENT. It should be observed that the choice of $x(0)$ determines not only which solution we will obtain, but also how many iterations it will take. In fact, in circuit applications a poor initial approximation can often result in *no convergence at all.* This will prove to be a major obstacle.

A Geometric Interpretation of Newton's Method

Now that we know the mechanics of Newton's method, let us consider what makes it work.



$$\tan\alpha = f'(x(0)) = \frac{f(x(0))}{\Delta x} \quad \Rightarrow \quad f'(x(0))\,(x(0) - x(1)) = f(x(0)) \quad \Rightarrow$$

$$\Rightarrow \quad f'(x(0))\,(x(1) - x(0)) = -f(x(0))$$

Similarly

$$f'(x(1))\,(x(1)-x(2)) = f(x(1)) \quad \Rightarrow \quad f'(x(1))\,(x(2)-x(1)) = -f(x(1))$$

etc. This obviously corresponds to Newton's iterative scheme.

Extensions to Systems of Nonlinear Equations

So far we considered only nonlinear equation in one variable. What if we have something like

$$F(x) \equiv \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = 0$$

To generalize Newton's method to this kind of problem, we first need to recall the Taylor series expansion of such a function around some x^0 :

$$\begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} f_1(x_1^0, x_2^0) \\ f_2(x_1^0, x_2^0) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \; . \; . \; .$$

The term

$$J(x^0) \equiv \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix}$$

plays the role of *derivative at* x^0, and is referred to as the *Jacobian of function F*. Consequently, for this type of equation we can use the Jacobian in place of the derivative in Newton's method. The actual iterative scheme is given as

$$x(k+1) = x(k) - [J(x(k))]^{-1} F(x(k))$$

COMMENT. Note that now $x(k+1)$, $x(k)$ and $F(x(k))$ are 2 × 1 *vectors*, and $J(x(k))$ is a 2 × 2 *matrix.*

In the general case, we have a system of n nonlinear equations

$$F(x) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_n(x_1, \ldots, x_n) \end{bmatrix} = 0$$

and the Jacobian $J(x(k))$ will be an $n \times n$ matrix

$$J(x(k)) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix}$$

When the system is large, inverting such a matrix is highly undesirable. This problem can be avoided by rewriting each Newton iteration as a system of *sparse linear equations*

$$J(x(k))\,\Delta x = -F\,(x(k))$$

where

$$\Delta x \equiv x(k+1) - x(k)$$

Such a system can be solved using the standard sparse matrix techniques discussed earlier (even when the number of equations is large). Note however, that $J(x(k))$ and $F(x(k))$ *change in each iteration*, so in general it is necessary to solve several different linear equations before convergence is achieved.

EXAMPLE

Let us consider the system of equations

$$F(x) \equiv \begin{bmatrix} 3\,x_1 x_2 \;-\; x_2^{\,-2} - e^{\,x_1} \\ x_1^{\,2} x_2 \;-\; \cos x_1 \;-\; x_2 \end{bmatrix} = 0$$

The Jacobian in Newton's method is

$$J\,(x(k)) \;=\; \begin{bmatrix} 3x_2(k) - e^{\,x_1(k)} & 3x_1(k) + 2\,x_2(k)^{-3} \\ 2\,x_1(k)\,x_2(k) \;+ \sin x_1(k) & x_1(k)^2 \;-\; 1 \end{bmatrix}$$

As before, the obtained solution will depend on the choice of $x(0)$.

Case 1

The initial choice $x_1(0) = 1$; $x_2(0) = 1$ produces solution $x_1{*} = 1.1616685$; $x_2{*} = 1.1383094$ after 4 iterations.

Case 2

The initial choice $x_1(0) = 0$; $x_2(0) = 2$ produces solution $x_1{*} = -\ 0.43276$; $x_2{*} = -\ 1.117006295$ after 5 iterations.

EXAMPLE

Let us now look at another system of nonlinear equations

$$F(x) \equiv \begin{bmatrix} x_1^5 + 2\log x_1 - x_2 \\ 3x_1x_2 - x_2e^{x_2} \end{bmatrix} = 0$$

In this case, the Jacobian in Newton's method is

$$J(x(k)) = \begin{bmatrix} 5x_1^4(k) + 2x_1(k)^{-1} & -1 \\ 3x_2(k) & 3x_1(k) - e^{x_2(k)} - x_2(k)e^{x_2(k)} \end{bmatrix}$$

and the obtained solution again depends on the choice of $x(0)$.

Case 1

The initial choice $x_1(0) = 1$; $x_2(0) = 1$ produces solution $x_1^* = 1.0160218$; $x_2^* = 1.1145071$ after 5 iterations.

Case 2

The initial choice $x_1(0) = 5$; $x_2(0) = -1$ produces solution $x_1^* = 0.82554$; $x_2^* = 0$ after 12 iterations.

DC ANALYSIS OF NONLINEAR CIRCUITS

In any DC analysis, it is assumed that all capacitors are *open circuits*, and that inductors are *short circuits*.

As we saw earlier, if the circuit is *linear*, the equations that describe it have the general form

$$Gx - w = 0$$

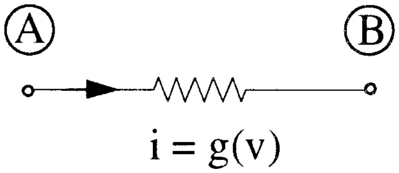
When the circuit has one or more *nonlinear* elements, the format becomes

$$Gx + p(x) - w = 0$$

where $p(x)$ is a nonlinear function.

Nonlinear resistors

A nonlinear resistor is a device where the current and voltage are not related by Ohm's law, but rather by some nonlinear function g:



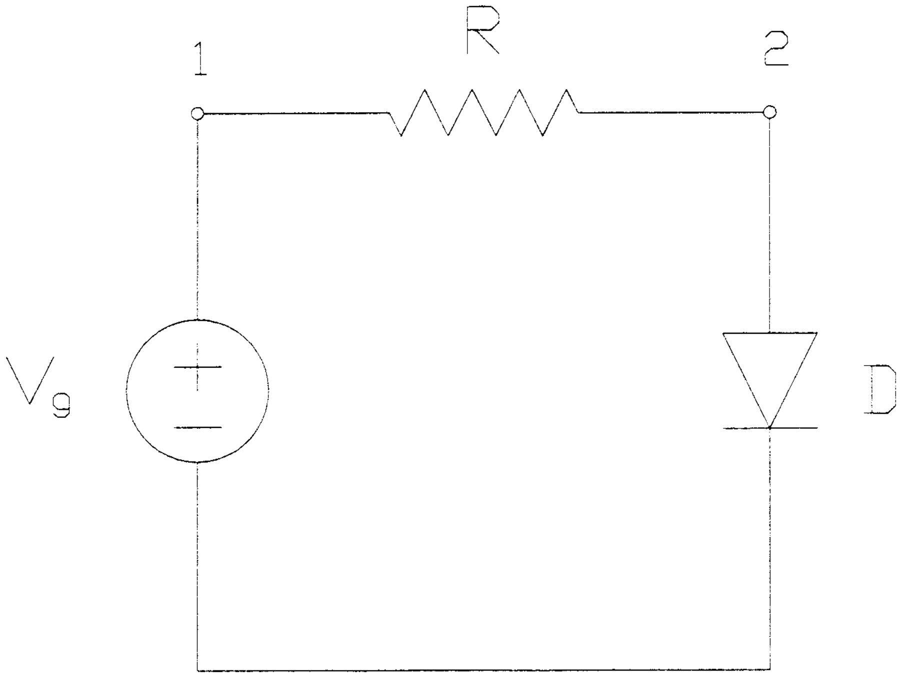
The stamp for such a device is

$$\begin{array}{l} A \;\ldots\; +i = g(V_A - V_B) \\[2em] B \;\ldots\; -i = -g(V_A - V_B) \end{array} \qquad \Rightarrow \qquad \begin{array}{c} A \\ B \end{array} \left[\begin{array}{c} g(V_A - V_B) \\ -g(V_A - V_B) \end{array} \right]$$

which implies that it contributes to $p(x)$ only.

EXAMPLE

A typical example of a nonlinear resistor is a *diode*. The following circuit illustrates how a diode affects the equation format.



Stamp for resistor (contribution to G only):

$$R: \quad \begin{array}{c|cc} & x_1 & x_2 \\ \hline 1 & \frac{1}{R} & -\frac{1}{R} \\ 2 & -\frac{1}{R} & \frac{1}{R} \end{array}$$

Stamp for voltage source (contribution to both G and w)

$$\begin{array}{c|cc} & x_1 & x_3 \\ \hline 1 & 0 & 1 \\ 3 & 1 & 0 \end{array} \quad ; \quad \begin{array}{c|c} 1 & 0 \\ 3 & V_g \end{array}$$

Stamp for nonlinear resistor (contribution to $p(x)$)

$$2 \quad \left[I_S\left(e^{\frac{x_2}{V_T}} - 1\right) \right]$$

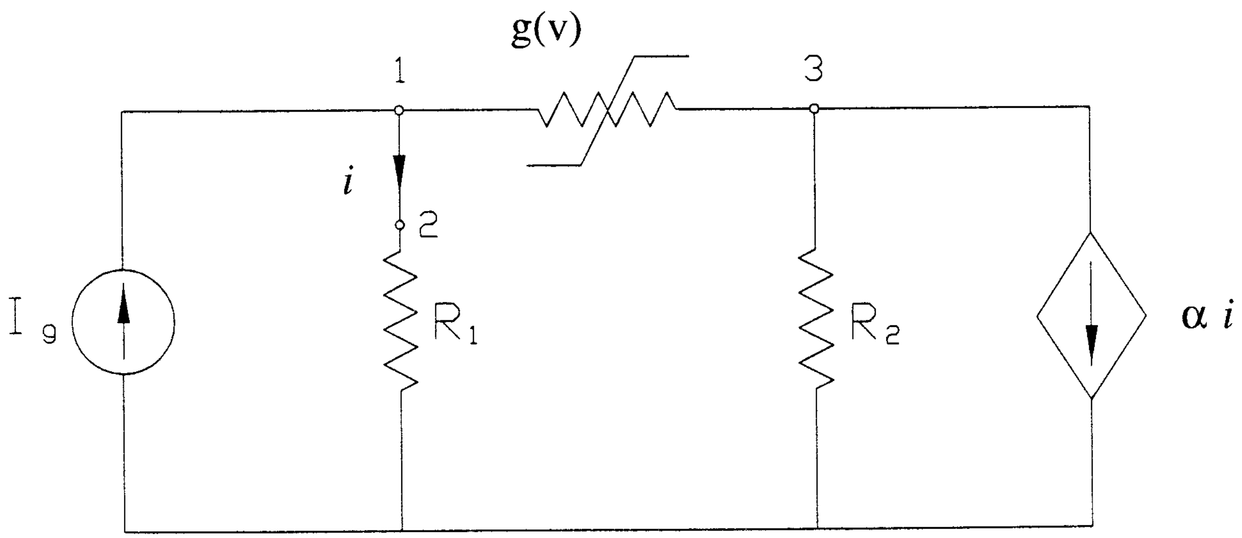
Combining all the stamps we obtain

$$\begin{array}{cc} & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} \frac{1}{R} & -\frac{1}{R} & 1 \\ -\frac{1}{R} & \frac{1}{R} & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ I_S\left(e^{\frac{x_2}{V_T}} - 1\right) \\ 0 \end{bmatrix} - \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} 0 \\ 0 \\ V_g \end{bmatrix} = 0$$

Obviously, these equations conform to the general format for nonlinear circuits.

EXAMPLE

In the following circuit, we consider a resistor with a *quadratic* type of nonlinearity, defined as $i = g(v) \equiv 3v^2$.



Stamps for resistors (contribution to G only):

$$R_1: \quad \begin{array}{c|c} & x_2 \\ \hline 2 & \frac{1}{R_1} \end{array} \quad ; \qquad R_2: \quad \begin{array}{c|c} & x_3 \\ \hline 3 & \frac{1}{R_2} \end{array}$$

Stamp for current source (contribution to w only):

$$I_g: \quad 1 \left[I_g \right]$$

Stamp for the controlled source (contibution to G only):

$$\begin{array}{c|cccc} & x_1 & x_2 & x_3 & x_5 \\ \hline 1 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & -1 \\ 3 & 0 & 0 & 0 & \alpha \\ 5 & 1 & -1 & 0 & 0 \end{array}$$

Stamp for the nonlinear resistor (contribution to $p(x)$ only):

$$\begin{array}{c} 1 \\ 3 \end{array} \begin{bmatrix} 3\,(x_1 - x_3)^2 \\ -3\,(x_1 - x_3)^2 \end{bmatrix}$$

Overall, we obtain

$$G = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{1}{R_1} & 0 & 0 & -1 \\ 0 & 0 & \frac{1}{R_2} & 0 & \alpha \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$p(x) = \begin{bmatrix} 3(x_1 - x_3)^2 \\ 0 \\ -3(x_1 - x_3)^2 \\ 0 \\ 0 \end{bmatrix} \quad ; \quad w = \begin{bmatrix} I_g \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and the equations again have the form

$$Gx + p(x) - w = 0$$

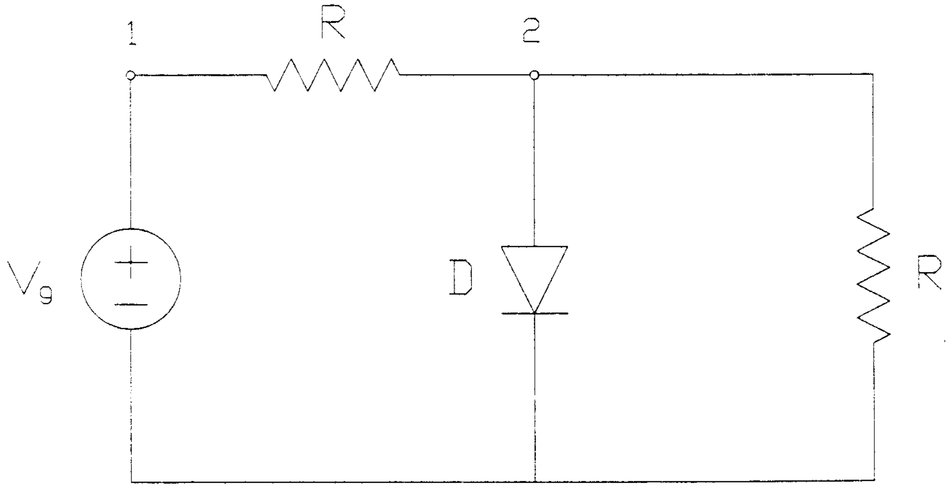
DC Analysis of Circuits With Diodes

Since the DC behavior of nonlinear circuits is described by a system of nonlinear algebraic equations, we can always use Newton's method to obtain a solution. In doing so, one of the most difficult problems is to find a good initial approximation (otherwise the iterative process may not converge at all).

In circuits where diodes are the only nonlinear elements, a good initial approximation can be obtained through an *approximate analysis* of the circuit. In such an analysis, we assume that a conducting diode has a voltage drop $V_D = 0.7V$, regardless of the current.

EXAMPLE

In this example we illustrate all the different stages of a DC analysis. Let us consider the circuit below, where $V_g = 5V$, $R = 1K$ and $I_S = 10^{-14}A$.



a) Circuit Equations

Stamps for resistors (contribution to *G* only)

$$R: \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} x_1 & \quad x_2 \end{array} \\ \begin{bmatrix} \frac{1}{R} & -\frac{1}{R} \\ -\frac{1}{R} & \frac{1}{R} \end{bmatrix} \end{array} \quad ; \quad R: \ 2 \begin{array}{c} x_2 \\ \begin{bmatrix} \frac{1}{R} \end{bmatrix} \end{array}$$

Stamp for voltage source (contribution to both G and w)

$$\begin{array}{cc} & \begin{matrix} x_1 & x_3 \end{matrix} \\ \begin{matrix} 1 \\ 3 \end{matrix} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array} ; \quad \begin{array}{cc} \begin{matrix} 1 \\ 3 \end{matrix} & \begin{bmatrix} 0 \\ V_g \end{bmatrix} \end{array}$$

Stamp for the diode (contribution to $p(x)$)

$$\begin{array}{cc} 2 & \left[I_S\left(e^{\frac{x_2}{V_T}} - 1\right) \right] \end{array}$$

Combining all the stamps we obtain

$$\begin{array}{cc} & \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} \frac{1}{R} & -\frac{1}{R} & 1 \\ -\frac{1}{R} & \frac{2}{R} & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ I_S\left(e^{\frac{x_2}{V_T}} - 1\right) \\ 0 \end{bmatrix} - \begin{array}{cc} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 0 \\ 0 \\ V_g \end{bmatrix} \end{array} = 0$$

b) Setting up Newton's Method

For the purposes of Newton's method, the circuit equations can be rewritten as

$$F(x) \equiv \begin{bmatrix} f_1(x_1, x_2, x_3) \\ f_2(x_1, x_2, x_3) \\ f_3(x_1, x_2, x_3) \end{bmatrix} = \begin{bmatrix} \frac{1}{R}x_1 - \frac{1}{R}x_2 + x_3 \\ -\frac{1}{R}x_1 + \frac{2}{R}x_2 + I_S(e^{\frac{x_2}{V_T}} - 1) \\ x_1 - V_g \end{bmatrix}$$

The Jacobian is then easily computed as

$$J(x(k)) = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} \frac{1}{R} & -\frac{1}{R} & 1 \\ -\frac{1}{R} & \left(\frac{2}{R} + \frac{I_S}{V_T} e^{\frac{x_2(k)}{V_T}}\right) & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

We should point out that this Jacobian can actually be expressed as

$$J(x(k)) = G + \frac{\partial p}{\partial x}$$

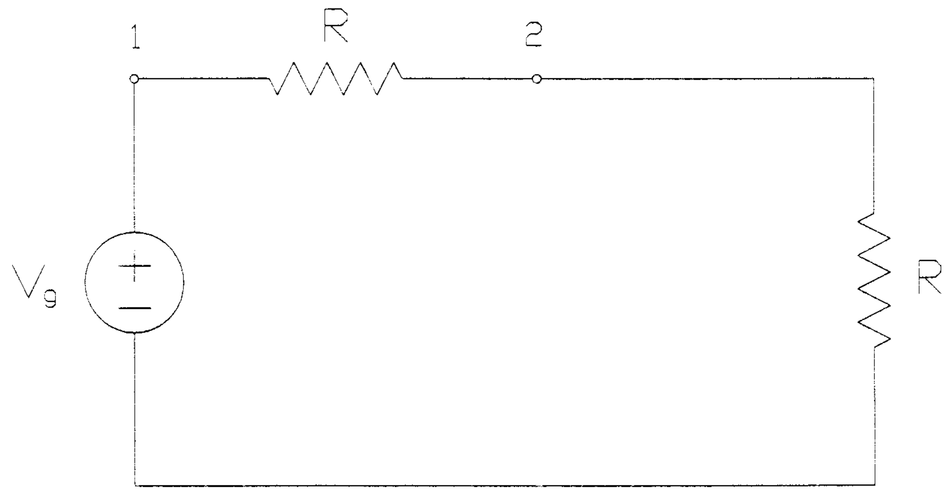
In other words, it consists of a constant part, G, and the term $\partial p/\partial x$ (which represents the Jacobian of $p(x)$).

The term $\partial p/\partial x$ can be formed from the *individual stamps* of each nonlinear element in the circuit. For example, in this circuit we have a diode current $I_D(x_2)$ contributing to equation $f_2(x)$; since this current depends only on x_2, it will contribute a term $\partial I_D/\partial x_2$ to element J(2, 2) of the overall Jacobian.

c) *Obtaining a Good Initial Approximation*

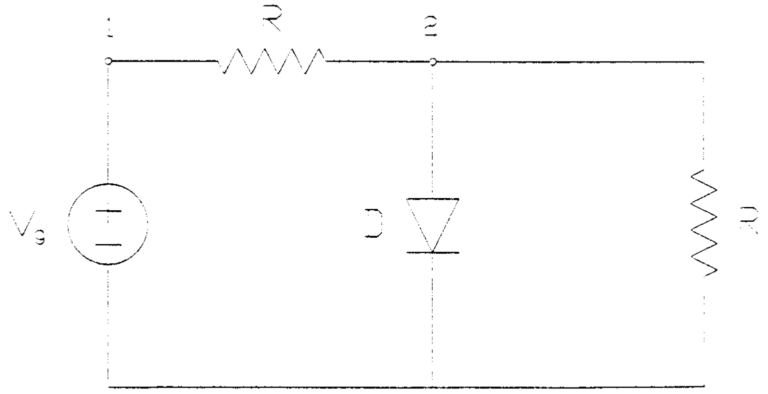
To get a good initial approximation, we first need to perform a simplified analysis of the circuit.

ASSUME DIODE IS OFF



In this case it follows that $V_D = 2.5$ V, which is clearly a contradiction (a diode voltage should not exceed +0.7V under any circumstances). Consequently, we conclude that this *assumption is incorrect.*

ASSUME DIODE IS ON



In this case we obtain $V_1 = 5$V; $V_2 = 0.7$V; $i = 4.3$mA. Since there are no contradictions, it follows that our *assumption was correct*, and that the DC solution can be estimated as:

$$x(0) = \begin{bmatrix} 5 \\ 0.7 \\ -4.3 \times 10^{-3} \end{bmatrix}$$

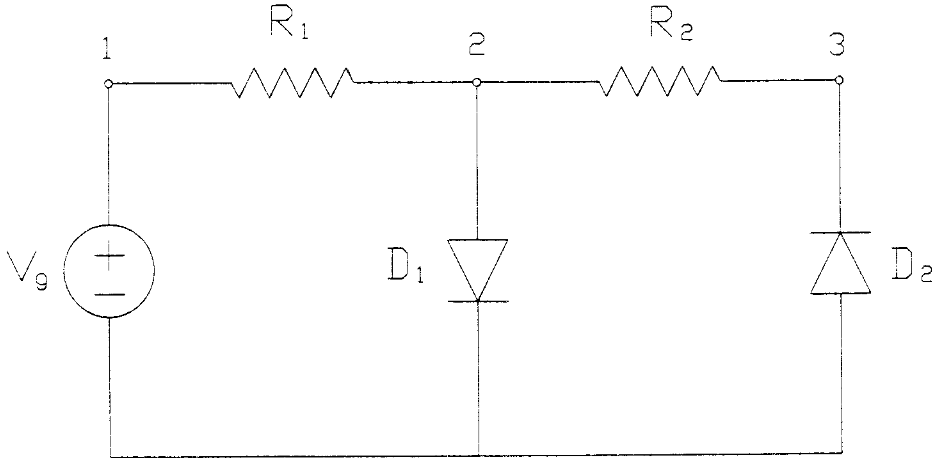
Using this as an initial approximation, Newton's method converges to:

$$x^* = \begin{bmatrix} 5.00000 \\ 0.67096 \\ -4.33 \times 10^{-3} \end{bmatrix}$$

after 6 iterations. This is the *exact* DC solution for our circuit.

EXAMPLE

In this example, we consider the DC analysis of a circuit with more than one diode.



We will assume that the diodes are identical (with $I_S = 10^{-14}A$), $V_g = 2V$ and $R_1 = R_2 = 1K$.

a) Circuit Equations

Stamps for resistors (contribution to *G* only):

$$R_1: \begin{array}{c|cc} & x_1 & x_2 \\ \hline 1 & \frac{1}{R_1} & -\frac{1}{R_1} \\ 2 & -\frac{1}{R_1} & \frac{1}{R_1} \end{array} \quad ; \quad R_2: \begin{array}{c|cc} & x_2 & x_3 \\ \hline 2 & \frac{1}{R_2} & -\frac{1}{R_2} \\ 3 & -\frac{1}{R_2} & \frac{1}{R_2} \end{array}$$

Stamp for voltage source (contribution to both G and w)

$$V_g : \quad \begin{array}{c|cc} & x_1 & x_4 \\ 1 & 0 & 1 \\ 4 & 1 & 0 \end{array} \quad ; \quad \begin{array}{c|c} 1 & 0 \\ 4 & V_g \end{array}$$

Stamps for the diodes (contribution to $p(x)$)

$$2 \; \left[I_S \left(e^{\frac{x_2}{V_T}} - 1 \right) \right] \quad ; \quad 3 \; \left[- I_S \left(e^{-\frac{x_3}{V_T}} - 1 \right) \right]$$

It is easily seen that

$$G = \begin{bmatrix} \frac{1}{R_1} & -\frac{1}{R_1} & 0 & 1 \\ -\frac{1}{R_1} & (\frac{1}{R_1} + \frac{1}{R_2}) & -\frac{1}{R_2} & 0 \\ 0 & -\frac{1}{R_2} & \frac{1}{R_2} & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

b) *Setting up Newton's Method*

The function $F(x)$ can be expressed as

$$\begin{bmatrix} f_1(x_1, x_2, x_3, x_4) \\ f_2(x_1, x_2, x_3, x_4) \\ f_3(x_1, x_2, x_3, x_4) \\ f_4(x_1, x_2, x_3, x_4) \end{bmatrix} = \begin{bmatrix} \frac{1}{R_1} x_1 - \frac{1}{R_1} x_2 + x_4 \\ - \frac{1}{R_1} x_1 + (\frac{1}{R_1} + \frac{1}{R_2}) x_2 - \frac{1}{R_2} x_3 + I_S (e^{\frac{x_2}{V_T}} - 1) \\ - \frac{1}{R_2} x_2 + \frac{1}{R_2} x_3 - I_S (e^{-\frac{x_3}{V_T}} - 1) \\ x_1 - V_g \end{bmatrix}$$

and we know once again that the Jacobian will consist of two terms:

$$J(x(k)) = G + \frac{\partial p}{\partial x}$$

To form $\partial p/\partial x$, we can consider the separate contribution of each diode in the form of a stamp.

i) Diode current $I_{D1}(x_2)$ appears in equation $f_2(x)$; since this current depends only on x_2, it will contribute a term $\partial I_{D1}/\partial x_2$ to element J(2, 2) of the overall Jacobian.

ii) Diode current - $I_{D2}(x_3)$ appears in equation $f_3(x)$; since this current depends only on x_3, it will contribute a term - $\partial I_{D2}/\partial x_3$ to element J(3, 3) of the overall Jacobian.

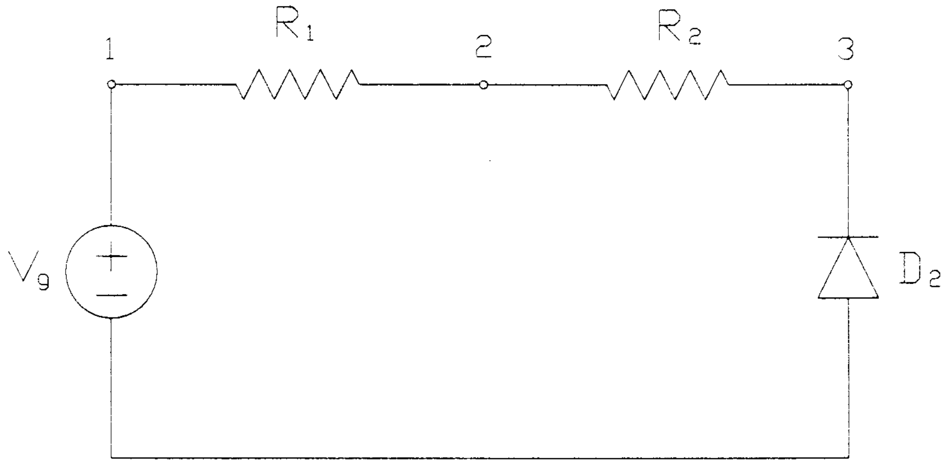
As a result,

$$\frac{\partial p}{\partial x} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \dfrac{I_S}{V_T} e^{\frac{x_2(k)}{V_T}} & 0 & 0 \\ 0 & 0 & \dfrac{I_S}{V_T} e^{-\frac{x_3(k)}{V_T}} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

c) *Obtaining a Good Initial Approximation*

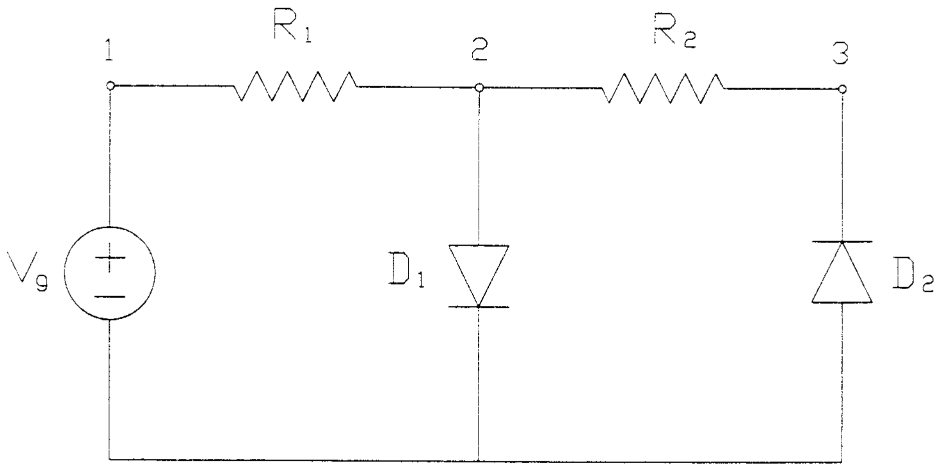
To get a good initial approximation, we first need to perform a simplified analysis of the circuit.

ASSUME D1 IS OFF AND D2 IS ON



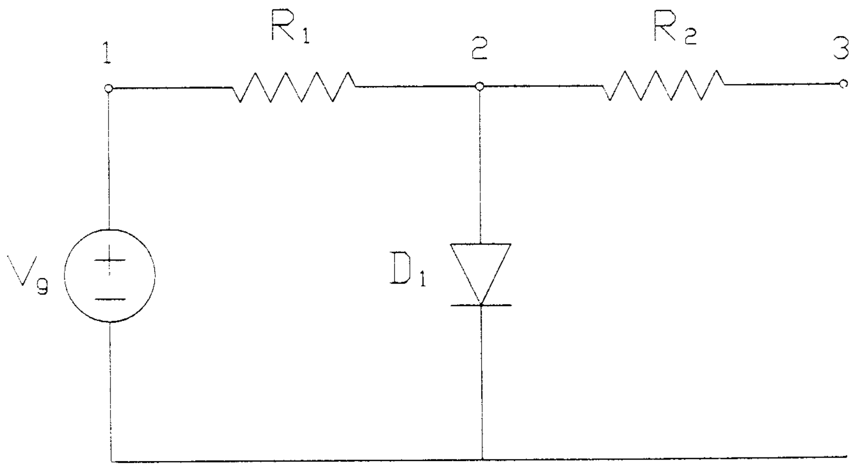
In this case, V_1 = 2V and V_3 = - 0.7V, implying that a *reverse* current of 1.35 mA flows through diode D2. This is a contradiction, so it follows that the assumption is *incorrect.*

ASSUME BOTH DIODES ARE ON



Here $V_2 = 0.7V$ and $V_3 = - 0.7V$, so a reverse current of 1.4 mA would flow through diode D2. This is again a contradiction, and the assumption must be *incorrect.*

ASSUME D1 IS ON AND D2 IS OFF



In this case, $V_1 = 2V$ and $V_2 = V_3 = 0.7V$; everything is consistent, so the assumption must be *correct.*

The DC solution can now be estimated as:

$$x(0) = \begin{bmatrix} 2 \\ 0.7 \\ 0.7 \\ -1.3 \times 10^{-3} \end{bmatrix}$$

Using this as an initial approximation, Newton's method converges to:

$$x^* = \begin{bmatrix} 2.00000 \\ 0.64591 \\ 0.64591 \\ -1.354 \times 10^{-3} \end{bmatrix}$$

after 7 iterations. This is the *exact* DC solution for our circuit.

DC Analysis of Circuits With Bipolar Transistors

The three-step method of DC analysis that was used in diode circuits can be extended to circuits with *bipolar tranistors* (BJT's). These devices are modeled by the Ebers - Moll equations

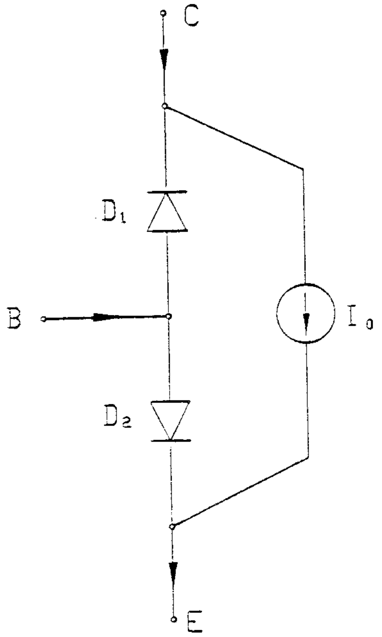
$$i_c = I_S\left[e^{\frac{V_{BE}}{V_T}} - e^{\frac{V_{BC}}{V_T}}\right] - \frac{I_S}{\beta_R}\left[e^{\frac{V_{BC}}{V_T}} - 1\right]$$

$$i_b = \frac{I_S}{\beta_F}\left[e^{\frac{V_{BE}}{V_T}} - 1\right] + \frac{I_S}{\beta_R}\left[e^{\frac{V_{BC}}{V_T}} - 1\right]$$

In addition, since $i_e = i_c + i_b$

$$i_e = I_S \left[e^{\frac{V_{BE}}{V_T}} - e^{\frac{V_{BC}}{V_T}} \right] + \frac{I_S}{\beta_F} \left[e^{\frac{V_{BE}}{V_T}} - 1 \right]$$

Schematically, this model can be represented in terms of *two diodes* and a *nonlinear current source*



The two diodes have currents

$$I_{D1} = \frac{I_S}{\beta_R} \left(e^{\frac{V_{BC}}{V_T}} - 1 \right)$$

$$I_{D2} = \frac{I_S}{\beta_F} \left(e^{\frac{V_{BE}}{V_T}} - 1 \right)$$

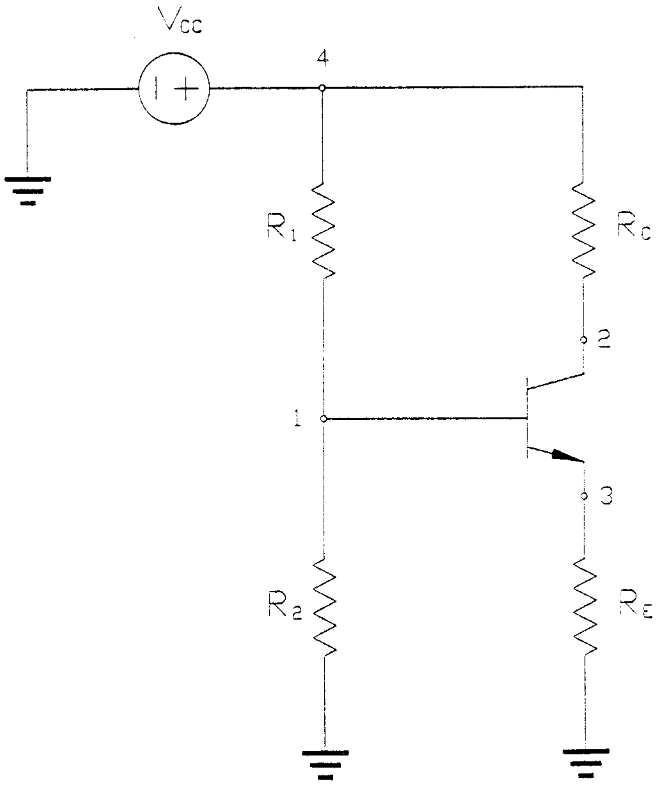
and the current source is defined as

$$I_0 = I_S\left(e^{\frac{V_{BE}}{V_T}} - e^{\frac{V_{BC}}{V_T}}\right)$$

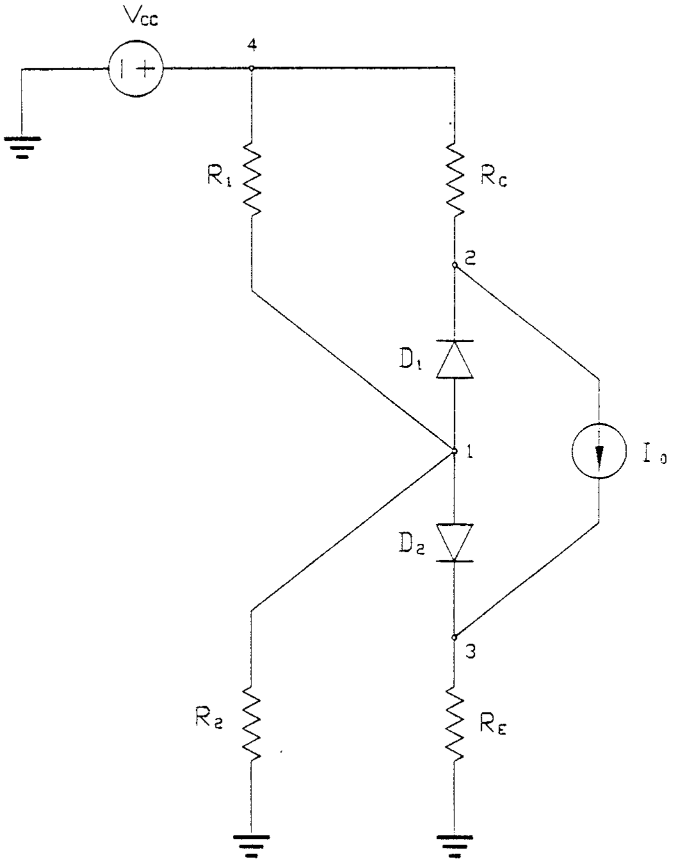
Note that in general this current source depends on *three* voltages - V_B,V_E and V_C.

EXAMPLE

In this example we will consider the DC analysis of a *common emitter amplifier*. For such an analysis, the coupling capacitors are *opened*, and the resulting circuit becomes



If we replace the bipolar transistor with its schematic model, we obtain



a) Circuit Equations

Stamps for resistors (contribution to G only):

$$R_1: \quad \begin{array}{c|cc} & x_1 & x_4 \\ 1 & \dfrac{1}{R_1} & -\dfrac{1}{R_1} \\ 4 & -\dfrac{1}{R_1} & \dfrac{1}{R_1} \end{array} \quad ; \quad R_2: \quad \begin{array}{c|c} & x_1 \\ 1 & \dfrac{1}{R_2} \end{array}$$

$$R_C: \quad \begin{array}{c|cc} & x_2 & x_4 \\ \hline 2 & \frac{1}{R_C} & -\frac{1}{R_C} \\ 4 & -\frac{1}{R_C} & \frac{1}{R_C} \end{array} \quad ; \quad R_E: \quad \begin{array}{c|c} & x_3 \\ \hline 3 & \frac{1}{R_E} \end{array}$$

Stamp for voltage source (contribution to both G and w)

$$V_g: \quad \begin{array}{c|cc} & x_4 & x_5 \\ \hline 4 & 0 & 1 \\ 5 & 1 & 0 \end{array} \quad ; \quad \begin{array}{c|c} 4 & 0 \\ 5 & V_g \end{array}$$

Stamps for the diodes (contribution to $p(x)$)

$$D_1: \quad \begin{array}{c|c} 1 & I_{DI}(x_1, x_2) \\ 2 & -I_{DI}(x_1, x_2) \end{array} \equiv \begin{array}{c|c} 1 & \frac{I_S}{\beta_R}\left(e^{\frac{x_1 - x_2}{V_T}} - 1\right) \\ 2 & -\frac{I_S}{\beta_R}\left(e^{\frac{x_1 - x_2}{V_T}} - 1\right) \end{array}$$

$$D_2: \quad \begin{array}{c} 1 \\ \\ 3 \end{array} \begin{bmatrix} I_{D2}(x_1, x_3) \\ \\ -I_{D2}(x_1, x_3) \end{bmatrix} \equiv \begin{array}{c} 1 \\ \\ 3 \end{array} \begin{bmatrix} \dfrac{I_S}{\beta_F}\left(e^{\frac{x_1 - x_3}{V_T}} - 1\right) \\ \\ -\dfrac{I_S}{\beta_F}\left(e^{\frac{x_1 - x_3}{V_T}} - 1\right) \end{bmatrix}$$

Stamp for the nonlinear current source (contribution to $p(x)$)

$$I_0: \quad \begin{array}{c} 2 \\ \\ 3 \end{array} \begin{bmatrix} I_0(x_1, x_2, x_3) \\ \\ -I_0(x_1, x_2, x_3) \end{bmatrix} \equiv \begin{array}{c} 2 \\ \\ 3 \end{array} \begin{bmatrix} I_S\left(e^{\frac{x_1 - x_3}{V_T}} - e^{\frac{x_1 - x_2}{V_T}}\right) \\ \\ -I_S\left(e^{\frac{x_1 - x_3}{V_T}} - e^{\frac{x_1 - x_2}{V_T}}\right) \end{bmatrix}$$

b) Setting up Newton's Method

Forming function $F(x)$ from the individual stamps is straightforward, and will not be shown here explicitly. The Jacobian, however, is more interesting and deserves some attention. In the following, we will focus on the contributions of nonlinear elements to term $\partial p/\partial x$.

i) Diode current $I_{D1}(x_1, x_2)$ appears in equations $f_1(x)$ and $f_2(x)$; since this current depends on both x_1 and x_2, it will contribute *four* terms to the overall Jacobian:

$$D_1: \quad \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc} \dfrac{\partial I_{D1}}{\partial x_1} & \dfrac{\partial I_{D1}}{\partial x_2} \\ -\dfrac{\partial I_{D1}}{\partial x_1} & -\dfrac{\partial I_{D1}}{\partial x_2} \end{array} \right] \end{array}$$

ii) Diode current $I_{D2}(x_1, x_3)$ appears in equations $f_1(x)$ and $f_3(x)$; since this current depends on both x_1 and x_3, it will contribute *four* terms to the overall Jacobian:

$$D_2: \quad \begin{array}{c} \\ 1 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} x_1 & x_3 \end{array} \\ \left[\begin{array}{cc} \dfrac{\partial I_{D2}}{\partial x_1} & \dfrac{\partial I_{D2}}{\partial x_3} \\ -\dfrac{\partial I_{D2}}{\partial x_1} & -\dfrac{\partial I_{D2}}{\partial x_3} \end{array} \right] \end{array}$$

iii) The nonlinear current source $I_0(x_1, x_2, x_3)$ appears in equations $f_2(x)$ and $f_3(x)$; since this current depends on x_1, x_2 and x_3, it will contribute *six* terms to the overall Jacobian (that is, it has a *rectangular* stamp):

$$I_0 : \begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline 2 & \frac{\partial I_0}{\partial x_1} & \frac{\partial I_0}{\partial x_2} & \frac{\partial I_0}{\partial x_3} \\ 3 & -\frac{\partial I_0}{\partial x_1} & -\frac{\partial I_0}{\partial x_2} & -\frac{\partial I_0}{\partial x_3} \end{array}$$

The overall Jacobian can now be formed in the usual way, as

$$J(x(k)) = G + \frac{\partial p}{\partial x}$$

c) *Obtaining a Good Initial Approximation*

As in the case of diode circuits, to get a good initial approximation for Newton's method we need to perform a simplified analysis of the circuit. We will use the following approximations for the bipolar transistor:

Active region

In this region, we assume that $V_{BE} = 0.7$ V, and that $i_C = \beta_F i_B$. Since β_F is typically ≥ 100, the base current is of the order of microamps and can be neglected where appropriate.

Cut off region

In this region, we assume $i_B = i_C = i_E = 0$ (that is, we can eliminate the transistor from the circuit).

Saturation

In this region, we assume that $V_{BE} = 0.7$ V and $V_{BC} = 0.4$ V.

For the common emitter amplifier, the transistor is designed to operate in the *active region*. Using the approximations for that region, we have

$$V_1 = \frac{R_2}{R_1 + R_2} V_{CC} \ ; \qquad V_3 = V_1 - 0.7 \ ; \qquad I_E = \frac{V_3}{R_E}$$

and also

$$I_C = \frac{\beta_F}{1 + \beta_F} I_E \ ; \qquad V_2 = V_{CC} - R_C I_C$$

If we assume that $\beta_F = 100$, and use the element values from the previous section, the DC solution can be estimated as:

$$x(0) = \begin{bmatrix} 4 \\ 6 \\ 3.3 \\ 12 \\ -1.99 \times 10^{-3} \end{bmatrix}$$

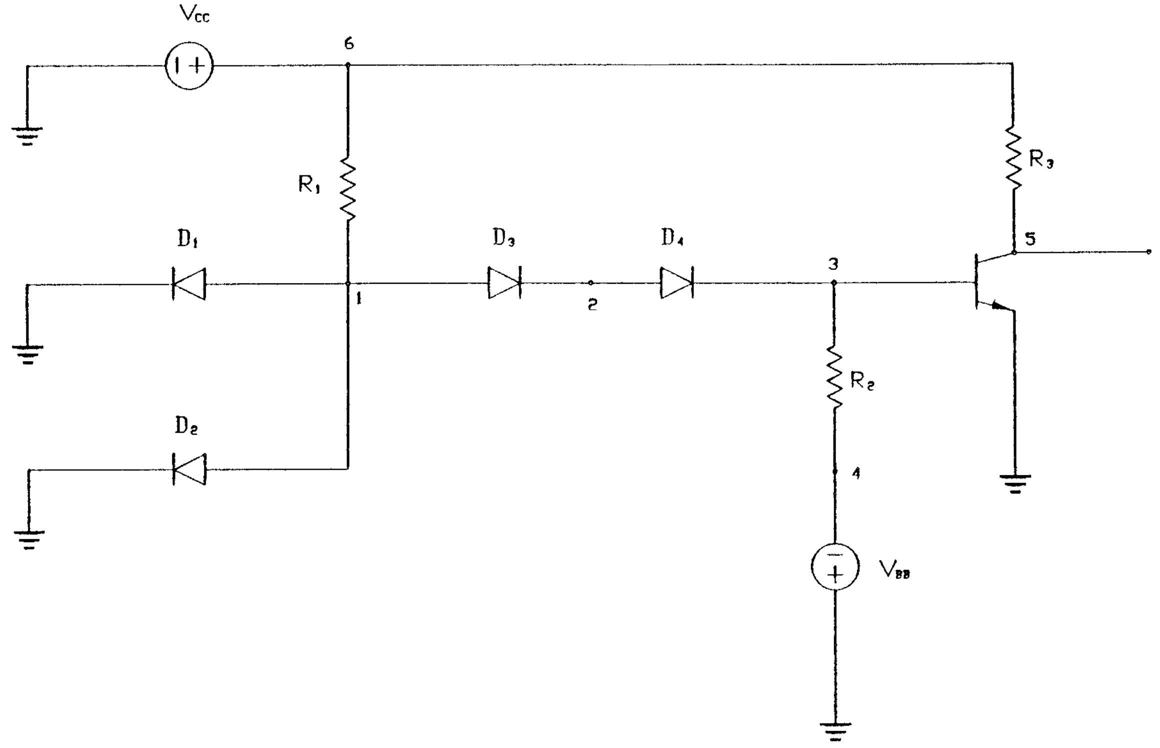
Using this as an initial approximation, Newton's method converges to:

$$x^* = \begin{bmatrix} 3.9742 \\ 6.2021 \\ 3.2208 \\ 12.000 \\ -1.97 \times 10^{-3} \end{bmatrix}$$

after 11 iterations.

EXAMPLE

In the previous example it was fairly easy to obtain a good initial approximation for Newton's method, largely because we had advance knowledge of the region in which the transistor should operate. We now provide an example where this is much more difficult to do:

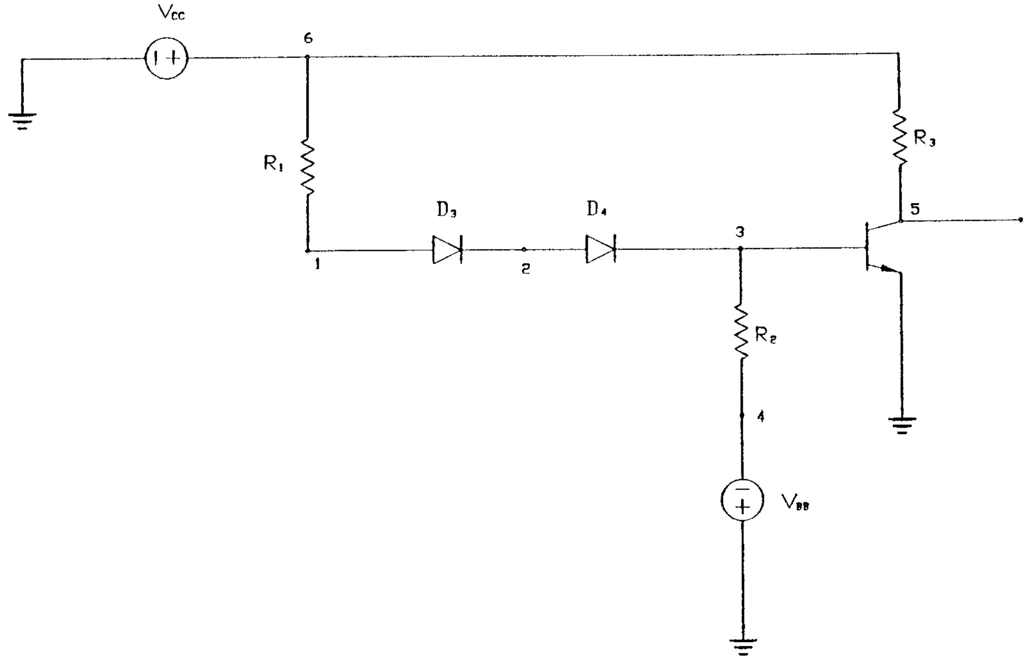


This circuit represents a DTL (diode transistor logic) NAND gate, and the element values are:

R_1 = 2K; R_2 = 5K; R_3 = 4K; V_{CC} = 4V; V_{BB} = 2V.

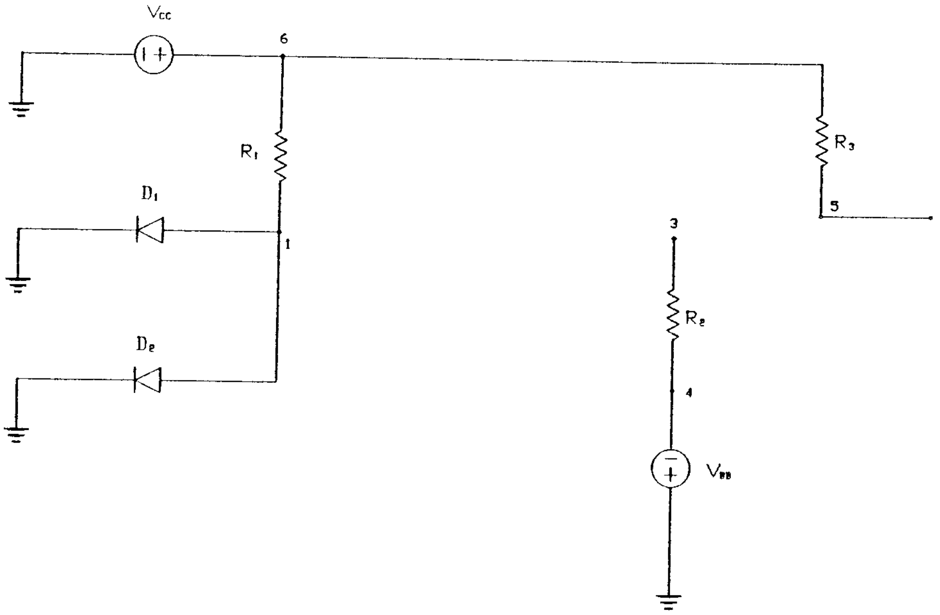
Here we have *five* nonlinear devices, so guessing their mode of operation involves numerous combinations. In the interest of brevity, we will consider only a few of them (including, of course, the correct one).

ASSUME D1, D2 - OFF; D3, D4 - ON; BJT - ACTIVE



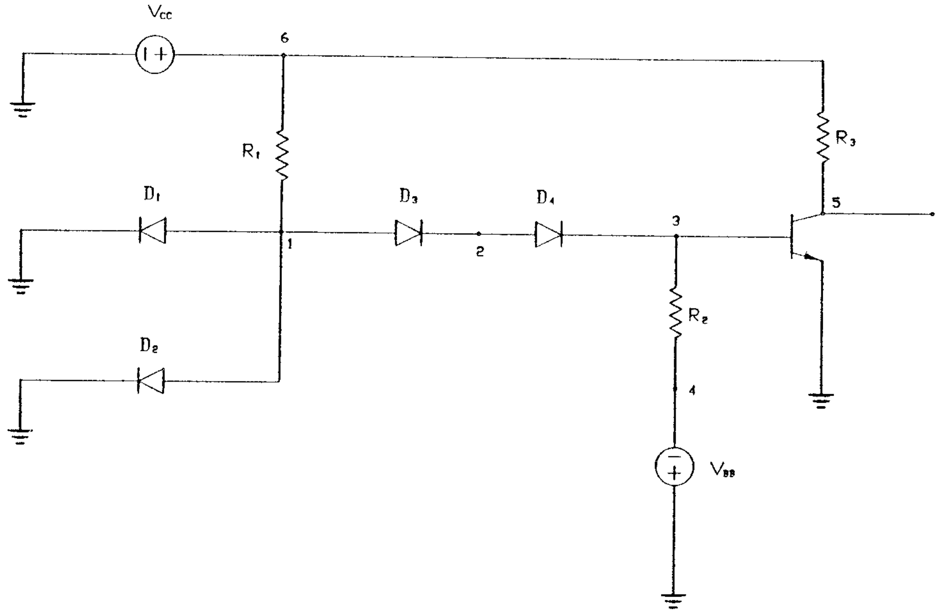
In this case it follows that V_1 = 2.1 V, which is a contradiction since diodes D1 and D2 are assumed to be off (besides, a diode voltage should not exceed + 0.7V under any circumstances). Therefore, this assumption is *incorrect.*

ASSUME D1, D2 - ON; D3, D4 - OFF; BJT - OFF



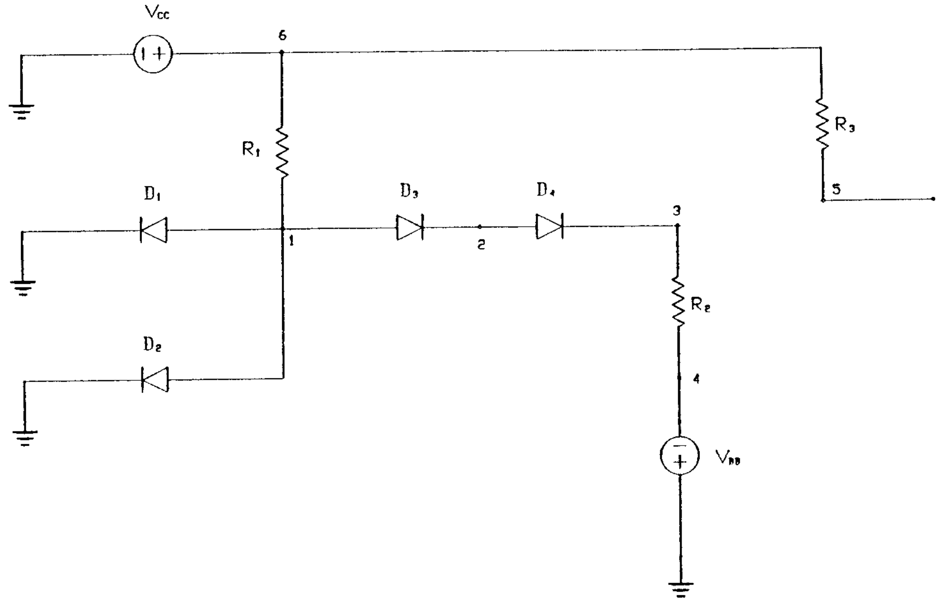
In this case it follows that V_1 = 0.7 V and V_3 = - V_{BB} = - 2V. This implies that there is a 2.7 V drop across diodes D3 and D4, which is impossible (the maximal positive drop would be 1.4 V, in case both D3 and D4 are conducting). As a result, this assumption is *incorrect.*

ASSUME D1, D2, D3, D4 - ON; BJT - SATURATED



In this case, the four conducting diodes imply that $V_1 = 0.7$ V, $V_2 = 0$ V and $V_3 = -0.7$ V. However, this means that $V_{BE} = -0.7$ V, which is *inconsistent* with the assumption that the transistor is saturated.

<u>ASSUME D1, D2, D3, D4 - ON; BJT - OFF</u>



In this case there are no contradictions, so the assumption is *correct*. The DC solution can then be estimated as:

$$x(0) = \begin{bmatrix} 0.7 \\ 0 \\ -0.7 \\ -2 \\ 4 \\ 4 \\ -1.65 \times 10^{-3} \\ -0.26 \times 10^{-3} \end{bmatrix}$$

SECTION IV:

TRANSIENT ANALYSIS

EQUATIONS FOR TRANSIENT ANALYSIS

In transient analysis, it becomes necessary to consider inductors and capacitors. As a result, the circuit will now be described by *differential equations*. The general format of these equations is

$$Ex' + Gx + p(x) - w = 0$$

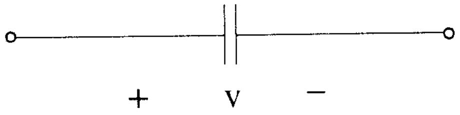
We begin by considering stamps for capacitors and inductors.

Capacitors

Capacitors are characterized by a relationship between the *charge* and the *voltage*. We will confine our discussion to *linear* capacitors, where

$$q = C V$$

The contribution of a linear capacitor to circuit equations is



$$A \ldots \quad q' = CV' \equiv C(V_A' - V_B')$$

$$B \ldots \quad -q' = -C(V_A' - V_B')$$

The corresponding stamp contributes to matrix E only

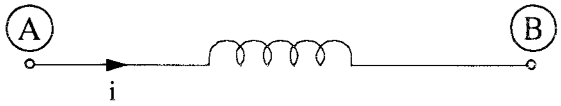
$$\begin{array}{c|cc|} & V_A' & V_B' \\ A & C & -C \\ B & -C & C \end{array}$$

Inductors

Inductors are characterized by a relationship between the *magnetic flux* and the *current*. For a *linear* inductor, this relationship is

$$\phi = L\,i$$

The contribution of a linear inductor to circuit equations is



$A \ldots \ +i$

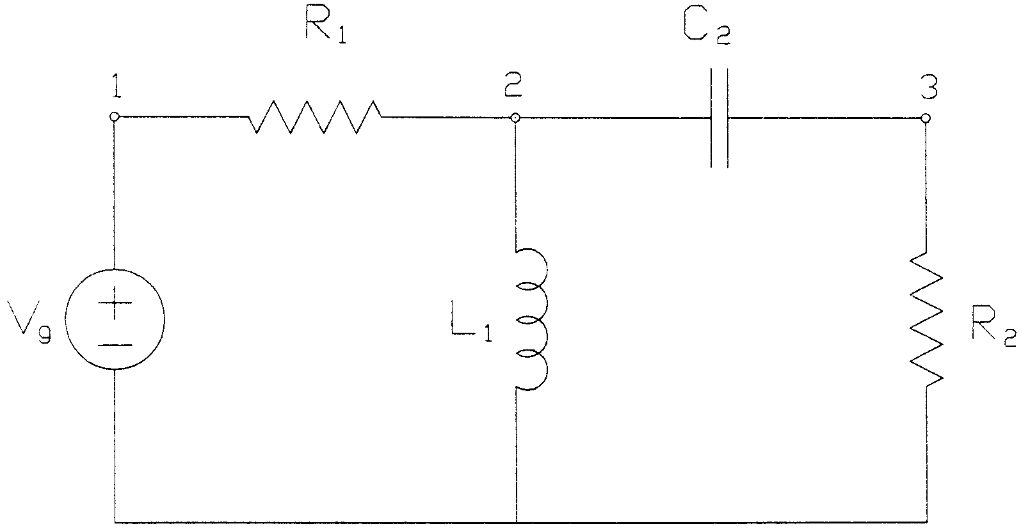
$B \ldots \ -i$

$COM: \quad V_A - V_B = \Phi' = L i'$

The corresponding stamp contributes to both E and G:

$$\begin{array}{c} A \\ B \\ COM \end{array} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & L \end{bmatrix} \begin{bmatrix} V_A' \\ V_B' \\ i' \end{bmatrix} ; \quad \begin{array}{c} A \\ B \\ COM \end{array} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} V_A \\ V_B \\ i \end{bmatrix}$$

EXAMPLE



Stamps for resistors (contribution to G only)

$$R_1 : \quad \begin{array}{c|cc} & x_1 & x_2 \\ \hline 1 & \frac{1}{R_1} & -\frac{1}{R_1} \\ 2 & -\frac{1}{R_1} & \frac{1}{R_1} \end{array} \quad ; \quad R_2 : \quad \begin{array}{c|c} & x_3 \\ \hline 3 & \frac{1}{R_2} \end{array}$$

Stamp for voltage source (contribution to both G and w)

$$\begin{array}{c|cc|} & x_1 & x_4 \\ 1 & 0 & 1 \\ 4 & 1 & 0 \end{array} \quad ; \quad \begin{array}{c|c|} 1 & 0 \\ 4 & V_g \end{array}$$

Stamp for the capacitor (contribution to E only)

$$C_1 : \quad \begin{array}{c|cc|} & x_2 & x_3 \\ 2 & C_1 & -C_1 \\ 3 & -C_1 & C_1 \end{array}$$

Stamp for the inductor (contribution to E and G)

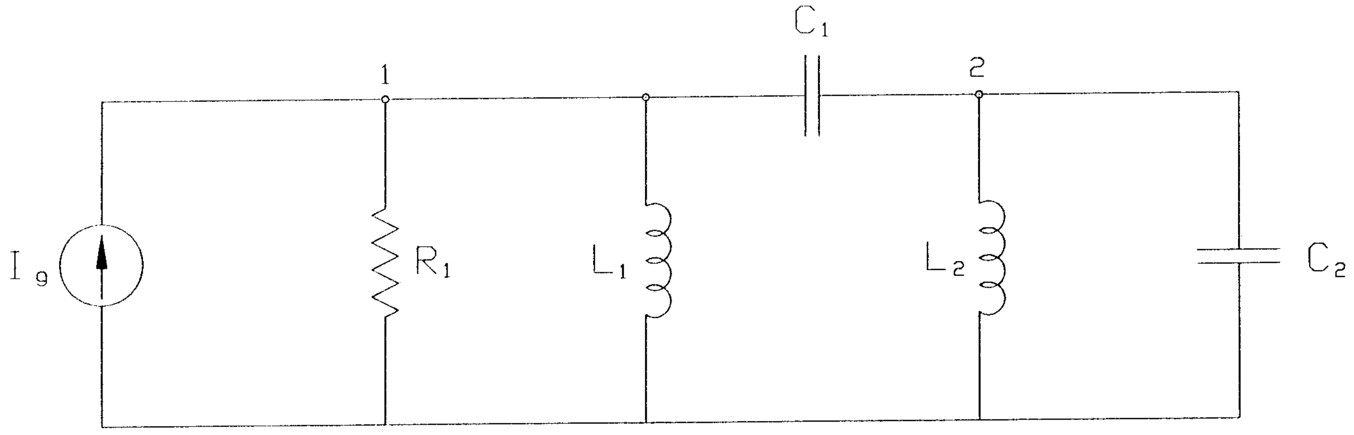
$$L_1 : \quad \begin{array}{c|cc|} & x_2 & x_5 \\ 2 & 0 & 0 \\ 5 & 0 & L_1 \end{array} \quad ; \quad \begin{array}{c|cc|} & x_2 & x_5 \\ 2 & 0 & 1 \\ 5 & -1 & 0 \end{array}$$

Overall, we have

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & C_1 & -C_1 & 0 & 0 \\ 0 & -C_1 & C_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & L_1 \end{bmatrix}$$

$$G = \begin{bmatrix} \frac{1}{R_1} & -\frac{1}{R_1} & 0 & 1 & 0 \\ -\frac{1}{R_1} & \frac{1}{R_1} & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{R_2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix} \quad ; \quad w = \begin{bmatrix} 0 \\ 0 \\ 0 \\ V_g(t) \\ 0 \end{bmatrix}$$

EXAMPLE



Stamps for resistors (contribution to *G* only)

$$R_1: \quad \begin{array}{c|c} & x_1 \\ \hline 1 & \dfrac{1}{R_1} \end{array}$$

Stamps for the capacitors (contribution to *E* only)

$$C_1: \quad \begin{array}{c|cc} & x_1 & x_2 \\ \hline 1 & C_1 & -C_1 \\ 2 & -C_1 & C_1 \end{array} \quad ; \qquad C_2: \quad \begin{array}{c|c} & x_2 \\ \hline 2 & C_2 \end{array}$$

Stamp for current source (contribution to w only)

$$I_g : \quad \begin{array}{cc} 1 & \begin{bmatrix} I_g \end{bmatrix} \end{array}$$

Stamps for the inductors (contribution to E and G)

$$L_1 : \quad \begin{array}{cc} & \begin{array}{cc} x_1 & x_3 \end{array} \\ \begin{array}{c} 1 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 \\ 0 & L_1 \end{bmatrix} \end{array} \quad ; \quad \begin{array}{cc} & \begin{array}{cc} x_1 & x_3 \end{array} \\ \begin{array}{c} 1 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{array}$$

$$L_2 : \quad \begin{array}{cc} & \begin{array}{cc} x_2 & x_4 \end{array} \\ \begin{array}{c} 2 \\ 4 \end{array} & \begin{bmatrix} 0 & 0 \\ 0 & L_2 \end{bmatrix} \end{array} \quad ; \quad \begin{array}{cc} & \begin{array}{cc} x_2 & x_4 \end{array} \\ \begin{array}{c} 2 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{array}$$

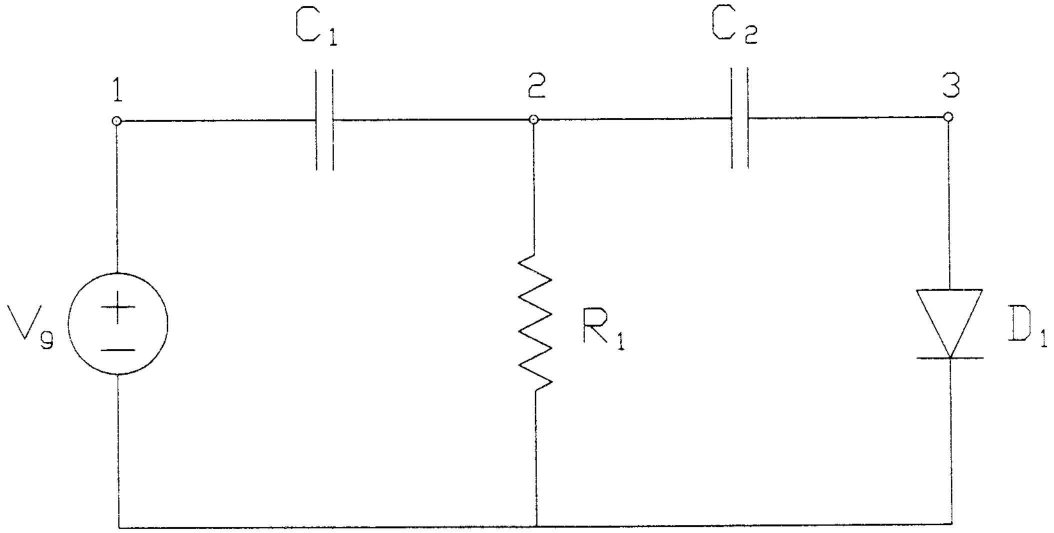
The overall matrices are

$$E = \begin{bmatrix} C_1 & -C_1 & 0 & 0 \\ -C_1 & (C_1 + C_2) & 0 & 0 \\ 0 & 0 & L_1 & 0 \\ 0 & 0 & 0 & L_2 \end{bmatrix}$$

and

$$G = \begin{bmatrix} \frac{1}{R_1} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad ; \quad w = \begin{bmatrix} I_g(t) \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

EXAMPLE



Stamp for the resistor (contribution to G only)

$$R_1: \quad \begin{array}{c|c} & x_2 \\ \hline 2 & \dfrac{1}{R_1} \end{array}$$

Stamps for the capacitors (contribution to E only)

$$C_1: \quad \begin{array}{c|cc} & x_1 & x_2 \\ \hline 1 & C_1 & -C_1 \\ 2 & -C_1 & C_1 \end{array} \quad ; \quad C_2: \quad \begin{array}{c|cc} & x_2 & x_3 \\ \hline 2 & C_2 & -C_2 \\ 3 & -C_2 & C_2 \end{array}$$

Stamp for the diode (contribution to $p(x)$ only)

$$D: \quad 3 \; \left[I_S \left(e^{\frac{x_3}{V_T}} - 1 \right) \right]$$

Stamp for voltage source (contribution to both G and w)

$$V_g: \quad \begin{array}{c} \\ 1 \\ 4 \end{array} \begin{array}{c} \begin{array}{cc} x_1 & x_4 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array} \quad ; \quad \begin{array}{c} 1 \\ 4 \end{array} \begin{bmatrix} 0 \\ V_g \end{bmatrix}$$

In this case, we obtain

$$E = \begin{bmatrix} C_1 & -C_1 & 0 & 0 \\ -C_1 & (C_1 + C_2) & -C_2 & 0 \\ 0 & -C_2 & C_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad ; \quad w = \begin{bmatrix} 0 \\ 0 \\ 0 \\ V_g \end{bmatrix}$$

and

$$G = \begin{bmatrix} \frac{1}{R_1} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad ; \quad p(x) = \begin{bmatrix} 0 \\ 0 \\ I_S\left(e^{\frac{x_3}{v_T}} - 1\right) \\ 0 \end{bmatrix}$$

COMMENT. It should be observed that the equations in this example have the general nonlinear format

$$E x' + Gx + p(x) - w(t) = 0$$

and that the last row of E has only zero elements. This means that one of the equations is *algebraic* (that is, contains no derivatives), and that the remaining three equations are *differential*. That type of situation is very common in the simulation of nonlinear circuits, and we refer to such mixed equations as *differential - algebraic equations* (*DAE*).

NUMERICAL INTEGRATION OF DAE

In the previous section, we pointed out that circuit equations

$$Ex' + Gx + p(x) - w(t) = 0$$

typically have a *singular* matrix E, which makes them differential-algebraic. The most general way of writing such equations is

$$f(x', x, t) = 0$$

To solve these equations numerically, we will consider a slightly simpler formulation

$$x' = f(x)$$

Numerical solution

Select a time step h, and a sequence of points t_0, $t_1 = t_0 + h$, $t_2 = t_1 + h$, ... , $t_n = t_{n-1} + h$, ... The equation must be satisfied in *all* these points, which implies

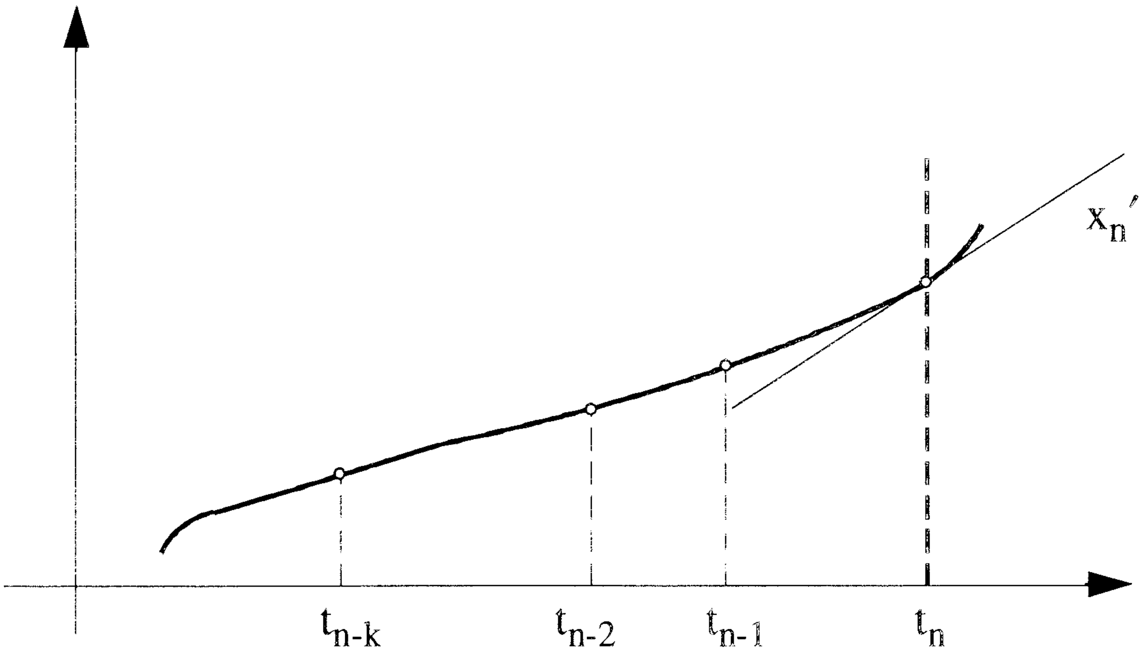
$$x'(t_n) = f(x(t_n))$$

The objective will be to approximate the derivative $x'(t_n)$ using $k + 1$ points x_n, x_{n-1}, ... , x_{n-k} as well as s ($s < k$) previously computed derivatives x'_{n-1}, ... , x'_{n-s} . We will say that this is an approximation of *order k.*

Since there are $k + s + 1$ known points we use an *interpolation polynomial* of order $m \equiv k + s$

$$x_m(t) = \sum_{i=0}^{m} d_i \left(\frac{t_n - t}{h} \right)^i = \alpha_0 + \alpha_1 t + \dots + \alpha_m t^m$$

The interpolation polynomial will be used to approximate x'_n with $x'_m(t_n)$.



By construction, the interpolation polynomial must satisfy

$$x_m(t_{n-j}) = x_{n-j} \qquad (j = 0, 1, \ldots, k)$$

$$x_m'(t_{n-j}) = x_{n-j}' \qquad (j = 1, 2, \ldots, s)$$

The first k conditions produce

$$x_n = x_m(t_n) = d_0$$

$$x_{n-1} = x_m(t_{n-1}) = \sum_{i=0}^{m} d_i \qquad (t_n - t_{n-1} = h)$$

$$x_{n-2} = x_m(t_{n-2}) = \sum_{i=0}^{m} d_i 2^i \qquad (t_n - t_{n-2} = 2h)$$

$$\vdots$$

$$x_{n-k} = x_m(t_{n-k}) = \sum_{i=0}^{m} d_i k^i \qquad (t_n - t_{n-k} = kh)$$

To use the remaining *s* conditions, first observe that

$$x_m'(t) = -\frac{1}{h}\sum_{i=0}^{m} i\, d_i \left(\frac{t_n - t}{h}\right)^{i-1} \equiv -\frac{1}{h}\sum_{i=1}^{m} i\, d_i \left(\frac{t_n - t}{h}\right)^{i-1}$$

These conditions can now be rewritten as

$$-h\, x_{n-1}' = -h\, x_m'(t_{n-1}) = \sum_{i=1}^{m} i\, d_i \qquad (t_n - t_{n-1} = h)$$

$$-h\, x_{n-2}' = -h\, x_m'(t_{n-2}) = \sum_{i=1}^{m} i\, d_i\, 2^{i-1} \qquad (t_n - t_{n-2} = 2h)$$

$$\vdots$$

$$-h\, x_{n-k}' = -h\, x_m'(t_{n-s}) = \sum_{i=1}^{m} i\, d_i\, s^{i-1} \qquad (t_n - t_{n-s} = sh)$$

We now have a total of $k + s + 1$ equations for unknown coefficients d_i.

In matrix form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots\cdots & 0 \\ 1 & 1 & 1 & & \cdots\cdots & 1 \\ 1 & 2 & 2^2 & & \cdots\cdots & 2^m \\ \vdots & \vdots & \vdots & & & \vdots \\ 1 & k & k^2 & & \cdots\cdots & k^m \\ 0 & 1 & 2 & & \cdots\cdots & m \\ 0 & 1 & 2\cdot 2 & & \cdots\cdots & m\,2^{m-1} \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 1 & 2s & & \cdots\cdots & m\,s^{m-1} \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_k \\ d_{k+1} \\ \vdots \\ \vdots \\ d_m \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ \vdots \\ x_{n-k} \\ -h\,x'_{n-1} \\ \vdots \\ \vdots \\ -h\,x'_{n-s} \end{bmatrix}$$

Observe that

$$x'_m(t) = -\frac{1}{h}\sum_{i=1}^{m} i\,d_i\left(\frac{t-t_n}{h}\right)^{i-1} = -\frac{1}{h}d_1 - \frac{1}{h}\sum_{i=2}^{m} i\,d_i\left(\frac{t-t_n}{h}\right)^{i-1}$$

and consequently

$$x'_n \approx x'_m(t_n) = -\frac{1}{h}\,d_1$$

This means that in each step we need to compute *only* d_1.

Computation of d_1

We have

$$d = V^{-1} z$$

and therefore

$$d_1 = [\ 0 \quad 1 \quad 0 \quad \ldots \quad 0\ ]\, d = e_1^T V^{-1} z$$

Denoting $\varphi_p \equiv e_1^{\mathrm{T}} \cdot V^{-1}$, it follows that $d_1 = \varphi_p z$ and

$$V^T \varphi_p^T = e_1$$

This is a system of linear equations in which φ_p^{T} is the unknown vector. It can be solved *off-line*, since both V and e_1 are *constant*. This is a big advantage, since computing the whole vector d would require solving a new system of equations in each step (the right hand side changes!).

Note that since

$$z = \begin{bmatrix} x_n \\ \vdots \\ x_{n-k} \\ -h\,x'_{n-1} \\ \vdots \\ -h\,x'_{n-s} \end{bmatrix}$$

it makes sense to partition φ_p accordingly, as

$$\varphi_p = [\ a_0\ \ a_1\ \ \ldots\ \ a_k \mid b_1\ \ \ldots\ \ b_s\ ]$$

Using this notation

$$-hx'_n \equiv d_1 = \sum_{j=0}^{k} a_j x_{n-j} - h \sum_{j=1}^{s} b_j x'_{n-j}$$

EXAMPLE 1

Approximate x'_n using x_n and x_{n-1}. In this case $k = 1$ and $m = 1$, which is a *first order approximation.* We have

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$$

and

$$V^T \varphi_p = e_1 \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{matrix} a_0 = -1 \\ a_1 = 1 \end{matrix}$$

The resulting approximation is known as the *backward Euler formula.*

$$-hx_n' = -x_n + x_{n-1} \quad \Rightarrow \quad x_n' = \frac{x_n - x_{n-1}}{h}$$

EXAMPLE 2

Approximate x'_n using x_n , $x_{n-1}$and x'_{n-1}. In this case $k = 1$ and $m = 2$, which is also a *first order approximation.* We have

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n-1} \\ -hx'_{n-1} \end{bmatrix}$$

and

$$V^T \varphi_p = e_1 \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} a_0 &= -2 \\ a_1 &= \ 2 \\ b_1 &= -1 \end{aligned}$$

Substituting the coefficients

$$-hx'_n = a_0 x_n + a_1 x_{n-1} - hb_1 x'_{n-1} = -2x_n + 2x_{n-1} + hx'_{n-1}$$

we obtain the *trapezoidal formula*

$$x'_n = \frac{2}{h}(x_n - x_{n-1}) - x'_{n-1}$$

EXAMPLE 3

Approximate x'_n using x_n , x_{n-1}, ... , x_{n-k} (and no derivatives). This is an *approximation of order k*, for which

$$\begin{bmatrix} 1 & 0 & 0 & \cdots\cdots & 0 \\ 1 & 1 & 1 & \cdots\cdots & 1 \\ 1 & 2 & 2^2 & \cdots\cdots & 2^k \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & k & k^2 & \cdots\cdots & k^k \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_k \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ \vdots \\ x_{n-k} \end{bmatrix}$$

and therefore

$$\begin{bmatrix} 1 & 1 & 1 & \cdots\cdots & 1 \\ 0 & 1 & 2 & \cdots\cdots & k \\ 0 & 1 & 2^2 & \cdots\cdots & k^2 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & 2^k & \cdots\cdots & k^k \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

In this case $m = k$, so

$$-hx_n' = \sum_{j=0}^{k} a_j x_{n-j} \quad \Rightarrow \quad x_n' = -\frac{1}{h}\sum_{j=0}^{k} a_j x_{n-j}$$

This corresponds to the class of *backward difference formulas* (also known as *Gear's formulas*).

Solution using the backward Euler formula

In this case our differential equation can be approximated as

$$\frac{x_n - x_{n-1}}{h} = f(x_n) \qquad n = 1, \ldots$$

To compute x_n, in each step we need to solve a *non-linear algebraic equation* of the form $F(x_n) = 0$, where

$$F(x_n) = -\frac{x_n}{h} + f(x_n) + \frac{x_{n-1}}{h}$$

and x_{n-1} is known. Note that given x_0, $x_0 \Rightarrow x_1$, $x_1 \Rightarrow x_2$, ... (that is, only x_0 is needed). Such a method is called *self-starting*.

APPLICATIONS OF TRANSIENT ANALYSIS

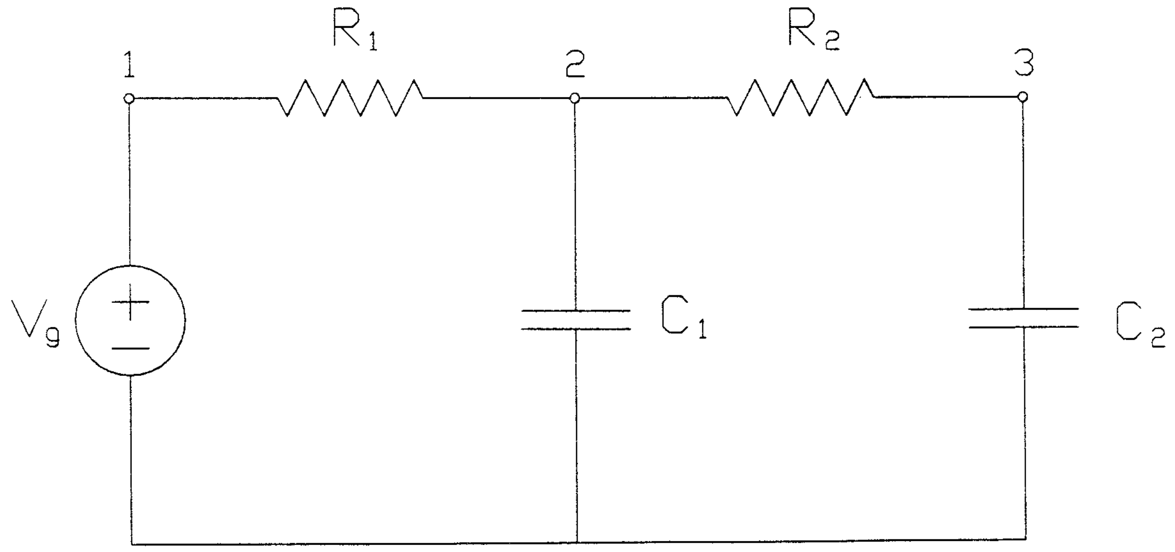
Transient analysis is widely used to simulate both analog and digital circuits. In recent years, there has been a great deal of interest in simulating large CMOS digital circuits, given that they are designed to operate at very high frequencies. In the following, we will consider this type of analysis in more detail.

The Step Response

The step response is of fundamental importance in circuit analysis. We will illustrate how it is computed by the following example.

EXAMPLE

Consider the circuit



in which $V_g(t)$ is a *unit step function.*

a) Circuit Equations

Stamps for resistors (contribution to G only):

$$R_1: \quad \begin{array}{c|cc} & x_1 & x_2 \\ 1 & \frac{1}{R_1} & -\frac{1}{R_1} \\ 2 & -\frac{1}{R_1} & \frac{1}{R_1} \end{array} \quad ; \quad R_2: \quad \begin{array}{c|cc} & x_2 & x_3 \\ 2 & \frac{1}{R_2} & -\frac{1}{R_2} \\ 3 & -\frac{1}{R_2} & \frac{1}{R_2} \end{array}$$

Stamp for voltage source (contribution to both G and w)

$$V_g: \quad \begin{array}{c|cc} & x_1 & x_4 \\ 1 & 0 & 1 \\ 4 & 1 & 0 \end{array} \quad ; \quad \begin{array}{c|c} 1 & 0 \\ 4 & V_g \end{array}$$

Stamps for capacitors (contribution to E)

$$C_1: \quad \begin{array}{c|c} & x_2 \\ 2 & C_1 \end{array} \quad ; \quad C_2: \quad \begin{array}{c|c} & x_3 \\ 3 & C_2 \end{array}$$

The overall matrices are

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & C_1 & 0 & 0 \\ 0 & 0 & C_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$G = \begin{bmatrix} \frac{1}{R_1} & -\frac{1}{R_1} & 0 & 1 \\ -\frac{1}{R_1} & (\frac{1}{R_1} + \frac{1}{R_2}) & -\frac{1}{R_2} & 0 \\ 0 & -\frac{1}{R_2} & \frac{1}{R_2} & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad ; \quad w = \begin{bmatrix} 0 \\ 0 \\ 0 \\ V_g(t) \end{bmatrix}$$

b) Discretization

It is first necessary to choose a step size, h. Once this is done, the derivative at time $t = t_n$ can be approximated as

$$x_n' = \frac{x_n - x_{n-1}}{h}$$

Substituting this into the circuit equations, we obtain

$$\frac{1}{h} E \, (x_n - x_{n-1}) + Gx - w(t_n) = 0$$

To compute x_n, we now need to solve

$$\left[\frac{1}{h} E + G\right] x_n = \frac{1}{h} E \, x_{n-1} - w(t_n)$$

Therefore, in *each* time point t_n, we must solve a *linear equation* to obtain x_n. This procedure continues until $t = t_{end}$ (starting from $x_0 = 0$).

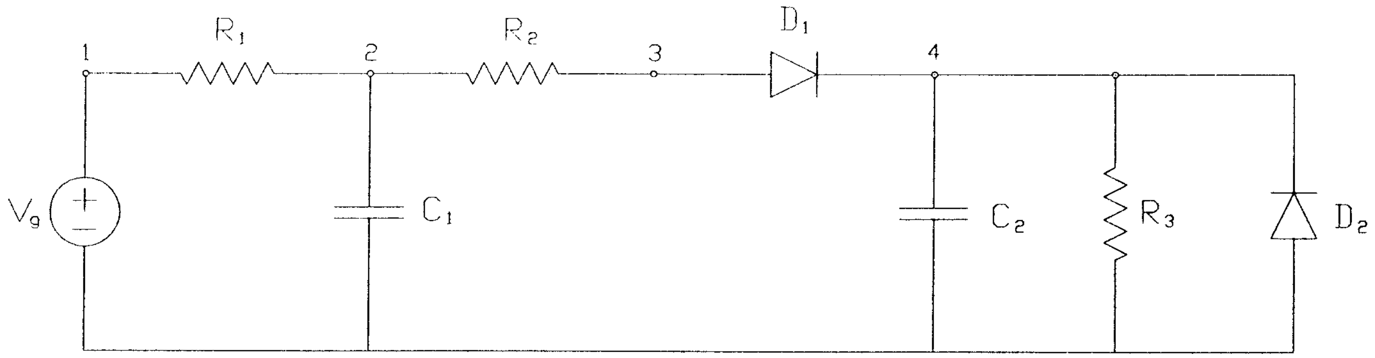
COMMENT. How does a computer handle a discontinuous function such as the unit step? The way to do this is to *approximate* an ideal step as a pulse with a short (but finite) rise time. It should also be noted that pulses in SPICE are always assumed to be *periodic*; consequently, to obtain the proper step response we should choose the period to be >> t_{end}.

Obtaining a Good Initial Approximation for DC Analysis

Another important application of transient analysis arises in the context of DC analysis. We saw earlier that for simple circuits with diodes and/or BJT's a good starting point for Newton's method can be obtained by various approximations. However, when the circuit becomes larger (or contains MOSFETs), this is no longer possible, and we need a general metod for determining an adequate $x(0)$. The following example illustrates how this can be accomplished using transient analysis.

EXAMPLE

Consider the nonlinear circuit shown below.



The element values are:

$R_1 = R_2 = 1K$; $R_3 = 2K$; $C_1 = 0.1nF$; $C_2 = 1pF$; $V_g = 5V$

and we will assume that $I_S = 10^{-14}$ A and $V_T = 25.2mV$.

Our objective in this case is to determine a good initial approximation for the DC solution *without* using any simplifications for the diodes. An obvious strategy would be to think of the voltage source as *5 volt step*, and to observe the transient response of the circuit for a sufficiently long time (that is, until the steady state is reached). At this point all voltages would clearly be close to their DC values, and could therefore be used as a good initial approximation for Newton's method. We now proceed to test the effectiveness of this approach.

a) Circuit Equations

Stamps for resistors (contribution to G only):

$$R_1: \begin{array}{c|cc} & x_1 & x_2 \\ \hline 1 & \frac{1}{R_1} & -\frac{1}{R_1} \\ 2 & -\frac{1}{R_1} & \frac{1}{R_1} \end{array} \quad ; \quad R_2: \begin{array}{c|cc} & x_2 & x_3 \\ \hline 2 & \frac{1}{R_2} & -\frac{1}{R_2} \\ 3 & -\frac{1}{R_2} & \frac{1}{R_2} \end{array}$$

$$R_3: \begin{array}{c|c} & x_4 \\ \hline 4 & \frac{1}{R_3} \end{array}$$

Stamps for capacitors (contribution to E)

$$C_1: \quad \begin{array}{c|c} & x_2 \\ 2 & \left[C_1 \right] \end{array} \quad ; \qquad C_2: \quad \begin{array}{c|c} & x_4 \\ 4 & \left[C_2 \right] \end{array}$$

Stamp for voltage source (contribution to both G and w)

$$V_g: \quad \begin{array}{c|cc} & x_1 & x_5 \\ 1 & 0 & 1 \\ 5 & 1 & 0 \end{array} \quad ; \qquad \begin{array}{c|c} 1 & 0 \\ 5 & V_g \end{array}$$

Stamps for diodes (contribution to $p(x)$)

$$D_1: \quad \begin{array}{c|c} 3 & I_S\left(e^{\frac{x_3 - x_4}{V_T}} - 1\right) \\ 4 & -I_S\left(e^{\frac{x_3 - x_4}{V_T}} - 1\right) \end{array} \quad ; \qquad D_2: \quad 4 \left[-I_S\left(e^{-\frac{x_4}{V_T}} - 1\right)\right]$$

The stamps can now be easily combined to form

$$Ex' + Gx + p(x) - w(t) = 0$$

b) Discretization

As in the previous example, we first need to choose a step size, h, and then approximate the derivative at time $t = t_n$ as

$$x_n' = \frac{x_n - x_{n-1}}{h}$$

Substituting this into the circuit equations, we have

$$F(x_n) \equiv \frac{1}{h} E\,(x_n - x_{n-1}) + Gx_n + p(x_n) - w(t_n) = 0$$

Therefore, in order to compute x_n we now need to solve a system of *nonlinear algebraic equations*. This means that we need to apply Newton's method in *every* time point, until $t = t_{end}$.

c) Setting Up Newton's Method

To begin with, let us simplify the notation and rename the vectors as: $x_n \equiv x$ and $x_{n-1} \equiv y$. By doing so, we avoid unnecessary subscripts, and also clearly distinguish the *unknown* vector x from the *known* vector y.

In light of the new notation, our equation becomes

$$F(x) \equiv \left[\frac{1}{h}E + G\right] x + p(x) - \frac{1}{h}Ey - w(t_n) = 0$$

For Newton's method, we have

$$F(x(k)) \equiv \left[\frac{1}{h}E + G\right] x(k) + p(x(k)) - \frac{1}{h}Ey - w(t_n)$$

and

$$J(x(k)) = \frac{1}{h}E + G + \frac{\partial p}{\partial x}$$

As before, the term $\partial p/\partial x$ can be formed from the contributions of individual nonlinear elements.

i) Diode current $I_{D1}(x_3, x_4)$ appears in equations $f_3(x)$ and $f_4(x)$; since this current depends on both x_3 and x_4, it will contribute *four* terms to the overall Jacobian:

$$D_1: \quad \begin{array}{cc} & \begin{matrix} x_3 & \quad x_4 \end{matrix} \\ \begin{matrix} 3 \\ \\ 4 \end{matrix} & \begin{bmatrix} \frac{\partial I_{D1}}{\partial x_3} & \frac{\partial I_{D1}}{\partial x_4} \\ -\frac{\partial I_{D1}}{\partial x_3} & -\frac{\partial I_{D1}}{\partial x_4} \end{bmatrix} \end{array}$$

ii) Diode current $I_{D2}(x_4)$ appears in equation $f_4(x)$ only; since this current depends on x_4, it will contribute *a single* term to the overall Jacobian:

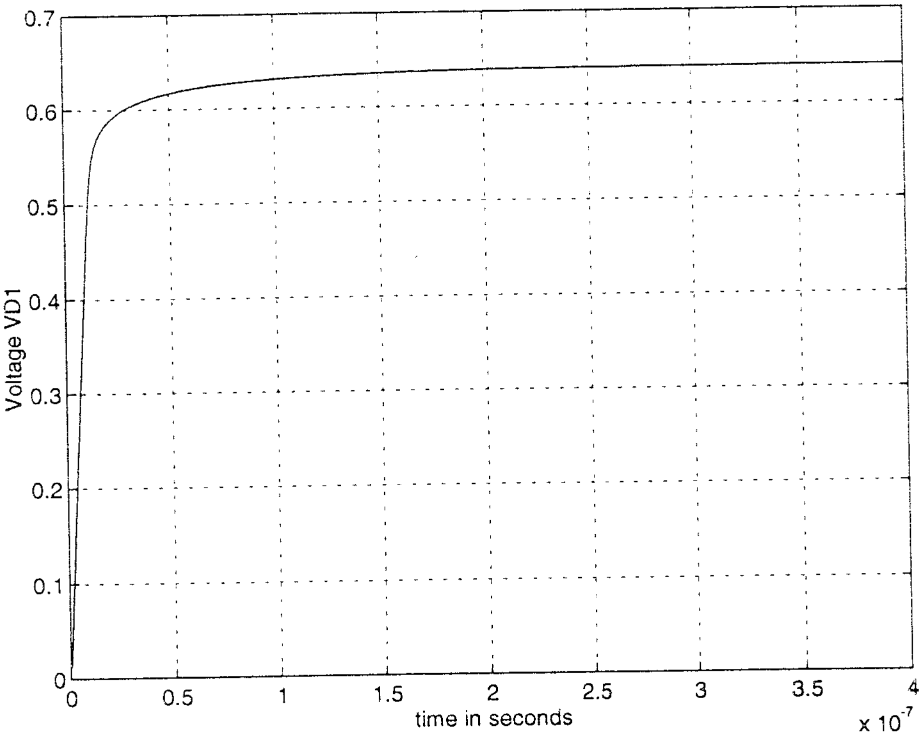
$$D_2: \quad \begin{array}{cc} & x_4 \\ 4 & \begin{bmatrix} -\frac{\partial I_{D2}}{\partial x_4} \end{bmatrix} \end{array}$$

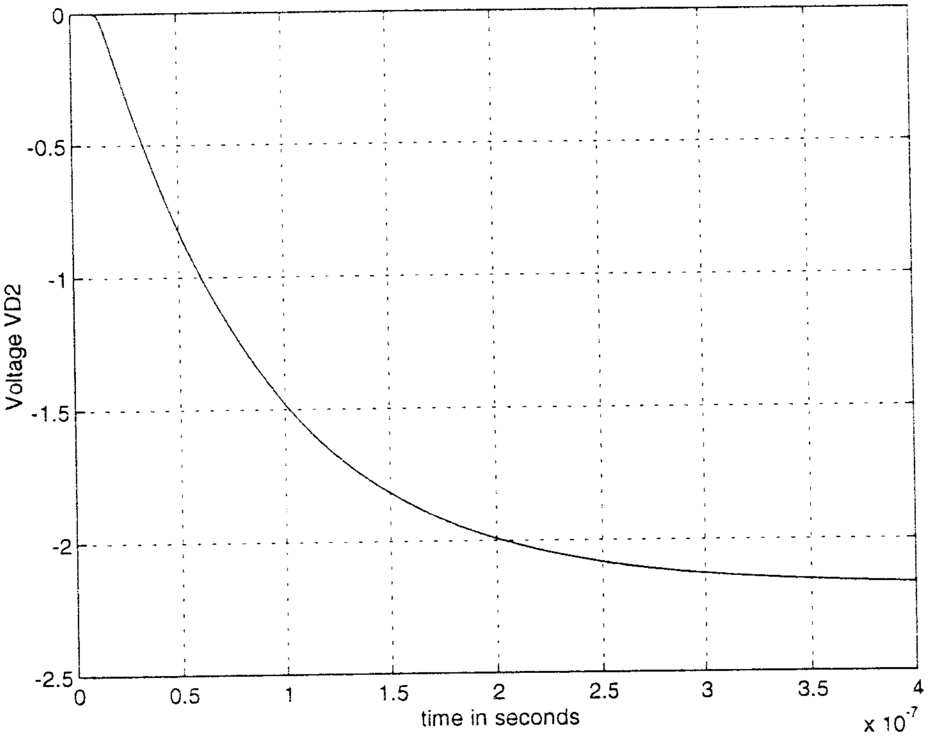
COMMENT. In applying Newton's method, it is usually a good idea to set $x(0) = y$ as the initial approximation. This is because h is typically very small, so x_n and x_{n-1} are not very different (as two successive time points).

d) *Simulation Results*

In order to perform the simulation, we need to treat the voltage source as a *5 volt step* and apply the described procedure in *each* time point. This generates a sequence of vectors $\{x_0, x_1, \ldots, x_n, \ldots\}$, which represent *discrete values* of $x(t)$.

In this example, an appropriate choice would be $h = 5 \times 10^{-10}$ and $t_{end} = 4 \times 10^{-7}$ seconds. The resulting diode voltages $V_{D1}(t)$ and $V_{D2}(t)$ are shown below.


0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Voltage VD1
0
0.5
1
1.5
2
2.5
3
3.5
4
time in seconds
x 10^{-7}


0
-0.5
-1
-1.5
-2
-2.5
Voltage VD2
0
0.5
1
1.5
2
2.5
3
3.5
4
time in seconds
x 10^{-7}

e) An Alternative Approach

The simulation shows that it takes about 400ns before the voltages come close to their steady state. Given that h = 0.5ns, it follows that we need to compute as many as 800 points, which is not efficient at all.

An alternative approach is to perform a much shorter simulation (perhaps only 40 or 50 points), since all we really want is a good initial approximation for the DC solution. In other words, it would make sense to set t_{end} = 20ns, as long as $x(t_{end})$ proves to be an adequate initial guess for Newton's method.

Using $x(t)$ evaluated at t_{end} = 20ns, we have the following initial guess

$$x(0) = \begin{bmatrix} 5.0000 \\ 0.8685 \\ 0.7568 \\ 0.1737 \\ -4.13 \times 10^{-3} \end{bmatrix}$$

With this $x(0)$ as the starting point for Newton's method, we obtain the DC solution after 12 iterations :

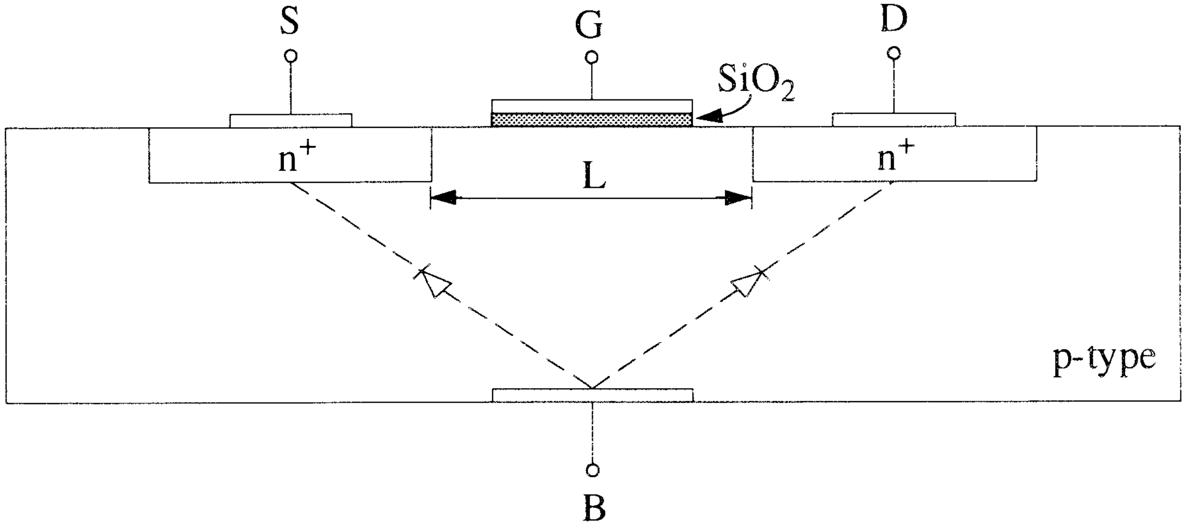
$$x^* = \begin{bmatrix} 5.0000 \\ 3.9101 \\ 2.8232 \\ 2.1827 \\ -1.09 \times 10^{-3} \end{bmatrix}$$

COMMENT. It is interesting to observe that in this case Newton's method will *not* converge from $x(0) = [0\ 0\ 0\ 0\ 0]^T$, or any other similar initial condition (such as *e. g.* $x(0) = [5\ 0\ 0\ 0\ 0]^T$). This obviously confirms the critical importance of obtaining a good initial condition.

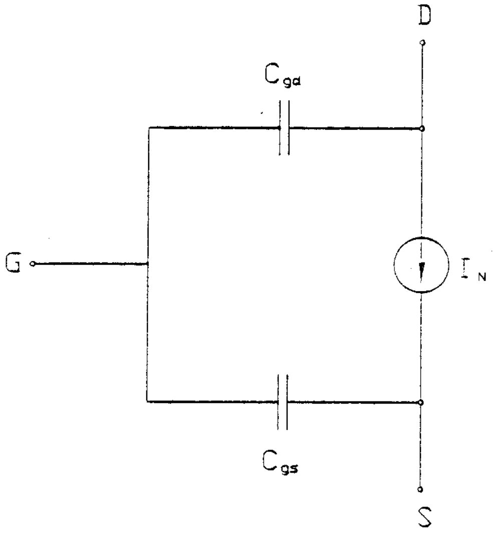
Digital CMOS Circuits

Transient analysis finds a major application in the design and simulation of digital circuits. Over the last ten years, transistor sizes have decreased dramatically, and the operating frequencies for many digital circuits have moved into the 100 MHz range. Under such conditions, it does not suffice to perform just a logic level analysis; the exact behavior of the circuit can be captured only through extensive transient simulation.

The most commonly used technology for digital circuits is CMOS. A schematic representation of an *n - channel enhancement type mosfet* is shown below.



The simplest model for this device consists of two capacitors and a nonlinear current source



where:

a) If $V_{GS} < V_{Tn}$

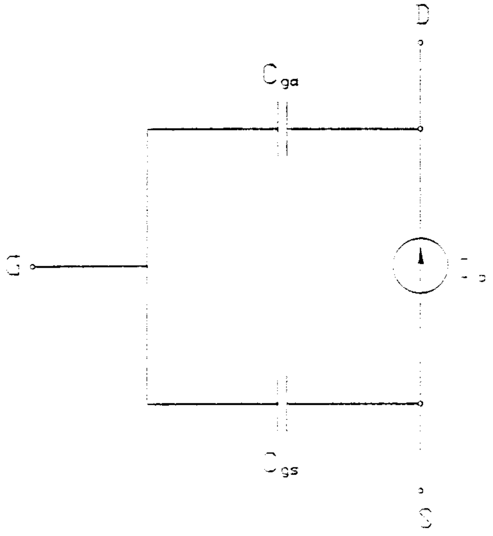
$$I_N = 0$$

b) If $V_{GS} \geq V_{Tn}$

$$I_N = \begin{cases} \frac{K}{2}(V_{GS} - V_{T_n})^2 & (V_{DS} \geq V_{GS} - V_{T_n}) \\ K\left[(V_{GS} - V_{T_n})V_{DS} - \frac{V_{DS}^2}{2}\right] & (V_{DS} < V_{GS} - V_{T_n}) \end{cases}$$

In the following, we will use $V_{Tn} = 1V$ and $K = 2 \times 10^{-5}\ A/V^2$, which are default values in SPICE.

The model for a *p - channel enhancement mosfet* is very similar to the previous one. In this case, we have



where:

a) If $V_{GS} > V_{Tp}$

$$I_P = 0$$

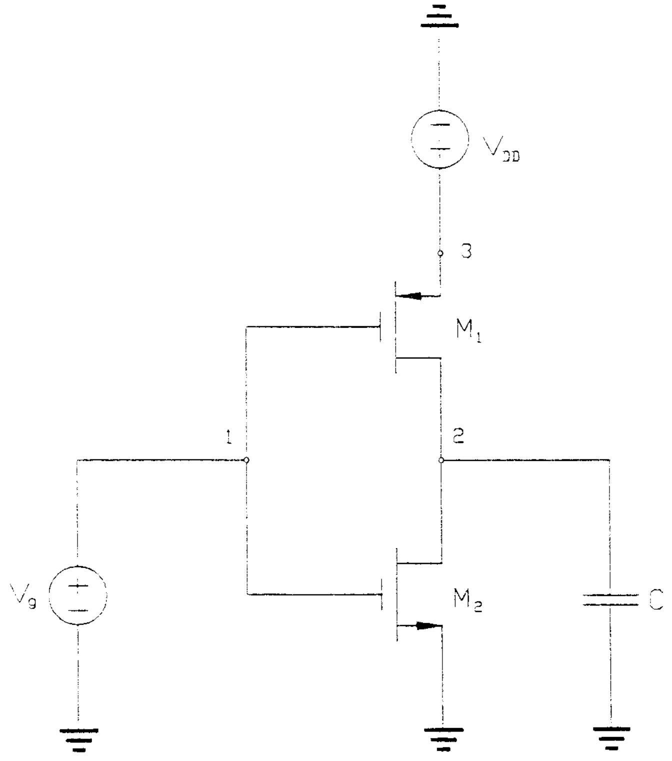
b) If $V_{GS} \leq V_{Tp}$

$$I_P = \begin{cases} \frac{K}{2} (V_{GS} - V_{T_p})^2 & (V_{DS} < V_{GS} - V_{T_p}) \\ K \left[(V_{GS} - V_{T_p}) V_{DS} - \frac{V_{DS}^2}{2} \right] & (V_{DS} \geq V_{GS} - V_{T_p}) \end{cases}$$

In the following, we will use $V_{Tp} = -1V$ and $K = 2 \times 10^{-5}$ A/V^2, which are again default values in SPICE.

EXAMPLE

The simplest CMOS logic gate is the *inverter*

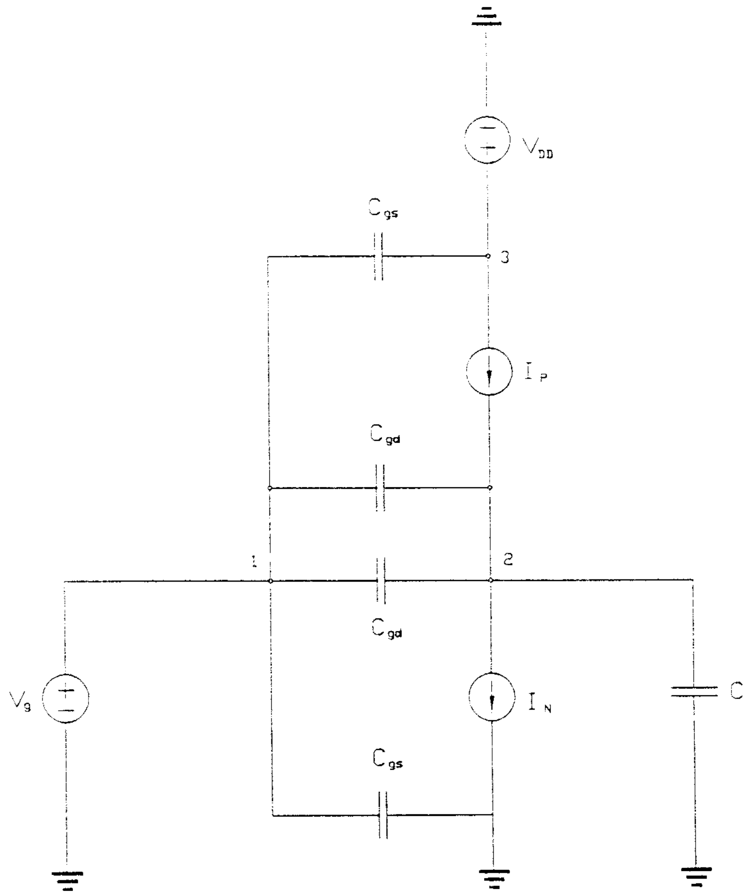


Note that this circuit has a *DC voltage source* (the power supply) and a *pulse source*. As a result, we first need to establish a DC operating point for the circuit, and then analyze the transient response to the incoming pulse.

To establish the DC operating point, we will use the same procedure as in the previous example. In other words, we will consider the transient response of the circuit to a *5 volt step*, and use this as the initial condition for Newton's method. The voltage of the pulse source will be set to *zero* in this process, and we will observe the transient response for 10 ns (this should provide an adequate initial approximation for the DC solution).

a) Circuit Equations

Given the models for n - channel and p - channel mosfets, the inverter circuit can be schematically represented as



Stamps for voltage sources (contribution to both *G* and *w*)

$$V_{DD}: \quad \begin{array}{c|cc} & x_3 & x_5 \\ 3 & 0 & 1 \\ 5 & 1 & 0 \end{array} \quad ; \quad \begin{array}{c|c} 3 & 0 \\ 5 & V_{DD} \end{array}$$

$$V_g: \quad \begin{array}{c|cc} & x_1 & x_4 \\ \hline 1 & 0 & 1 \\ 4 & 1 & 0 \end{array} \quad ; \quad \begin{array}{c|c} 1 & 0 \\ 4 & V_g \end{array}$$

Stamps for capacitors (contribution to E)

$$C_{gs}: \quad \begin{array}{c|cc} & x_1 & x_3 \\ \hline 1 & C_{gs} & -C_{gs} \\ 3 & -C_{gs} & C_{gs} \end{array} \quad ; \quad C_{gs}: \quad \begin{array}{c|c} & x_1 \\ \hline 1 & C_{gs} \end{array}$$

$$C_{gd}: \quad \begin{array}{c|cc} & x_1 & x_3 \\ \hline 1 & 2C_{gd} & -2C_{gd} \\ 3 & -2C_{gd} & 2C_{gd} \end{array} \quad ; \quad C_L: \quad \begin{array}{c|c} & x_2 \\ \hline 2 & C_L \end{array}$$

Stamps for the nonlinear current sources (contribution to $p(x)$)

$$I_N: \quad 1 \; [\, I_N \,]$$

$$I_P : \quad \begin{array}{c} 3 \\ \\ 2 \end{array} \left[\begin{array}{c} I_P \\ \\ -I_P \end{array} \right]$$

The stamps can again be easily combined to obtain the usual form

$$E x' + G x + p(x) - w(t) = 0$$

b) The DC Solution

After choosing a step size, h, we obtain the discretized equations as

$$F(x_n) \equiv \frac{1}{h} E\,(x_n - x_{n-1}) + G x_n + p(x_n) - w(t_n) = 0$$

For the purposes of DC analysis, we need to perform a transient analysis for $t \leq 10$ns. In each point, we can set $x_n \equiv x$ and $x_{n-1} \equiv y$, and rewrite the equation as

$$F(x) \equiv \left[\frac{1}{h} E + G \right] x + p(x) - \frac{1}{h} E y - w(t_n) = 0$$

Recalling that in DC analysis the voltage of the pulse source is *set to zero*, vector $w(t_n)$ becomes

$$w(t_n) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ V_{DD}(t_n) \end{bmatrix}$$

As before, the Jacobian in Newton's method has the form

$$J(x(k)) = \frac{1}{h}E + G + \frac{\partial p}{\partial x}$$

and $\partial p/\partial x$ can be formed from the contributions of the nonlinear current sources. Specifically,

i) Current source $I_P(x_1, x_2, x_3)$ appears in equations $f_2(x)$ and $f_3(x)$; since this current depends on both x_1, x_2 and x_3, it will contribute *six* terms to the overall Jacobian:

$$I_P : \quad \begin{array}{c} \\ 3 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} \quad x_1 \quad & \quad x_2 \quad & \quad x_3 \quad \end{array} \\ \begin{bmatrix} \dfrac{\partial I_P}{\partial x_1} & \dfrac{\partial I_P}{\partial x_2} & \dfrac{\partial I_P}{\partial x_3} \\ -\dfrac{\partial I_P}{\partial x_1} & -\dfrac{\partial I_P}{\partial x_2} & -\dfrac{\partial I_P}{\partial x_3} \end{bmatrix} \end{array}$$

ii) Current source $I_N(x_1, x_2)$ appears in equation $f_2(x)$ only; since this current depends on both x_1 and x_2, it will contribute *two* terms to the overall Jacobian:

$$I_N : \quad \begin{array}{cc} & \begin{array}{cc} x_1 & x_2 \end{array} \\ 2 & \left[\begin{array}{cc} \dfrac{\partial I_N}{\partial x_1} & \dfrac{\partial I_N}{\partial x_2} \end{array} \right] \end{array}$$

COMMENT. The models for both I_P and I_N depend on the mode of operation. Therefore, before the Jacobian is evaluated, we need to establish the region in which the mosfet is working at that point in time (that is, we need to determine if $V_G(t_n) > V_T$; $V_{DS}(t_n) > V_{GS}(t_n) - V_T$; etc.)

After performing a transient analysis for 10 ns, we obtain the following initial approximation for the DC solution

$$x(0) = \begin{bmatrix} 0.0000 \\ 4.3571 \\ 5.0000 \\ 3.75 \times 10^{-7} \\ -4.73 \times 10^{-5} \end{bmatrix}$$

Using this $x(0)$, we can solve the DC equations

$$Gx + p(x) - w = 0$$

bearing in mind that in these equations w is a *constant* vector

$$w = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}$$

After 4 iterations, Newton's method converges to the DC solution

$$x^* = \begin{bmatrix} 0 \\ 5 \\ 5 \\ 0 \\ 0 \end{bmatrix}$$

c) Transient analysis

Having obtained the DC operating point, we can now proceed to analyze the transient response to pulse source $V_g(t)$. We have already done most of the work for this step, since we previously performed a transient analysis for 10 ns to get a good initial condition for the DC solution.

There will be *three* important changes in this step.

(1) We need to include *both* the pulse source $V_g(t)$ *and* the DC source V_{DD} in vector w. In this case, V_{DD} will *not* be treated as a step function, but rather as a *constant* (that is, we will set $V_{DD} = 5$ V at all times).

(2) The initial vector in the transient analysis will be $x_0 = x^*$ instead of $x_0 = 0$. In other words, our transient analysis will *start from the DC operating point.*

(3) The simulation time t_{end} will be much larger than 10 ns.

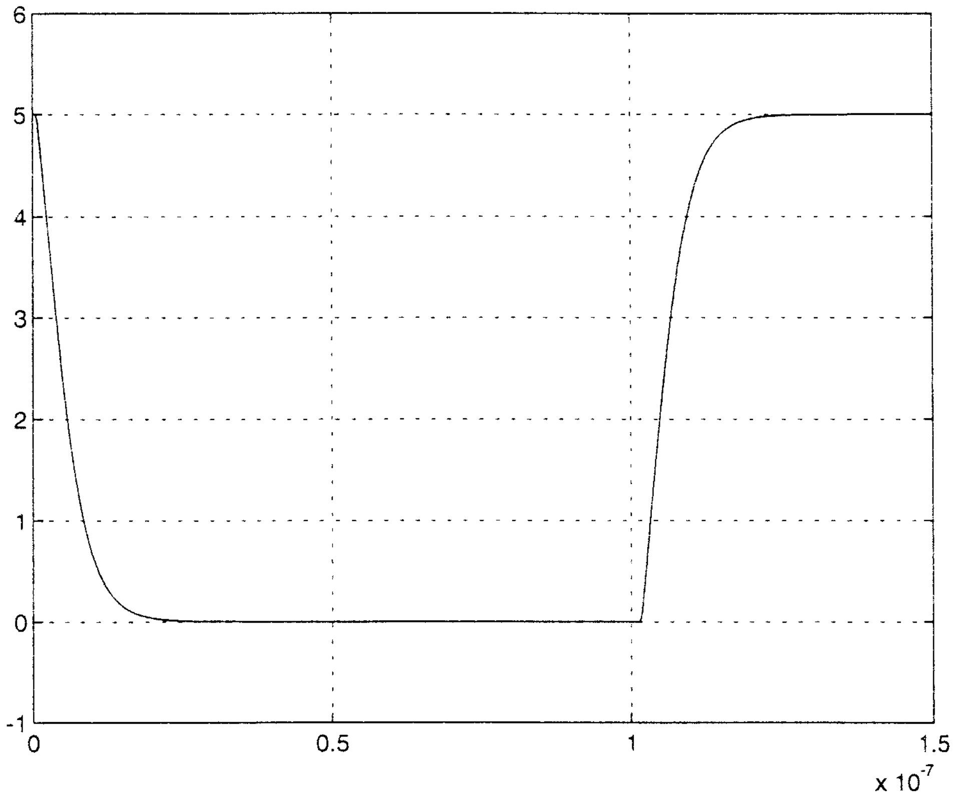
With these modifications in mind, we can proceed as before, solving the discretized equation in each step

$$F(x_n) \equiv \frac{1}{h} E\,(x_n - x_{n-1}) + Gx_n + p(x_n) - w(t_n) = 0$$

The Jacobian in Newton's method can be formed *exactly* as before.

To simulate how the inverter responds to an incoming pulse, we assumed that the n and p channel mosfets both have K = 2×10^{-5} A/V^2, with threshold voltages of $V_{Tn} = 1$V and $V_{Tp} = -1$ V, respectively. The *loading capacitor* was chosen as $C_L = 0.25$pF, and the *internal capacitances* were taken to be C_{gs} = 2fF and C_{gd} = 1fF.

W further assumed that $V_g(t)$ is a *5 volt pulse*, with $t_r = t_f$ = 1ns, and a *pulse width* of 100ns. With this in mind, the simulation was performed over a time of 150 ns, and the resulting output voltage is shown below.



SECTION V:

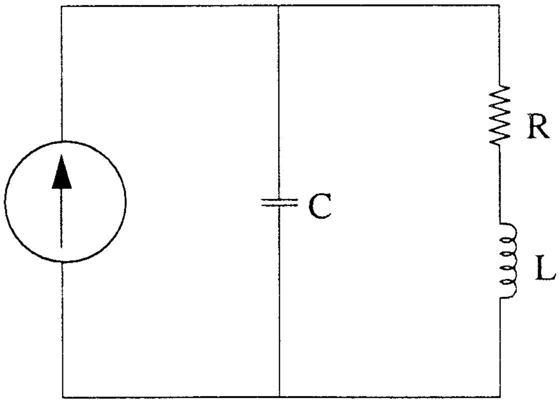
ADVANCED TOPICS IN TRANSIENT ANALYSIS

THE CHOICE OF INTEGRATION STEP

One of the most important issues in transient analysis is the choice of step *h*. The following example illustrates some of the issues that need to be considered in making this choice.

EXAMPLE

Consider the circuit below, in which R = 1K, C = 1pF and L = 1nH, and $I_g(t)$ is a *unit step function.*



The solutions for this circuit are

$$v_c(t) = A_{11} e^{-\frac{t}{\tau_1}} + A_{12} e^{-\frac{t}{\tau_2}} + R I_g$$

$$i_L(t) = A_{21} e^{-\frac{t}{\tau_1}} + A_{22} e^{-\frac{t}{\tau_2}} + I_g$$

with $\tau_1 = 10^{-9}$ s and $\tau_2 = 10^{-12}$ s. As a result, the solution has a *fast* *component* (corresponding to τ_2) and a *slow component* (corresponding to τ_1). Capturing the fast component would require a small step (*e. g.* $h = 5 \times 10^{-14}$). On the other hand, the slow component is "active" for at least 5ns, which implies a huge number of points with the original step h (10,000 points for a choice of $h = 5 \times 10^{14}$).

Circuits that exhibit this type of "two time-scale behavior" are frequently encountered in practice, and are referred to as *stiff circuits.* For such systems a fixed choice of h is obviously inadequate, and it becomes necessary to use a *variable step size.*

Numerical solution with a variable step size

Although a variable step size resolves the stiffness problem, it also raises a number of other issues. In particular, given a variable step size, are the previously developed approximations for x'_n still valid? The following analysis provides an answer to that question.

We begin by introducing notation $h(n) \equiv t_n - t_{n-1}$ and defining

$$\tau_j(n) \equiv \frac{t_n - t_{n-j}}{h(n)}$$

In that case, the interpolation polynomial becomes

$$x_m(t) = \sum_{i=0}^{m} d_i \left[\frac{t_n - t}{h(n)}\right]^i \equiv \sum_{i=0}^{m} d_i \tau^i(n)$$

As before,

$$x_n = x_m(t_n) = d_0$$

$$x_{n-1} = x_m(t_{n-1}) = \sum_{i=0}^{m} d_i$$

$$x_{n-2} = x_m(t_{n-2}) = \sum_{i=0}^{m} d_i \left[\frac{t_n - t_{n-2}}{h(n)} \right]^i \equiv \sum_{i=0}^{m} d_i \tau_2^i(n)$$

$$\vdots$$

$$x_{n-k} = x_m(t_{n-k}) = \sum_{i=0}^{m} d_i \left[\frac{t_n - t_{n-k}}{h(n)} \right]^i \equiv \sum_{i=0}^{m} d_i \tau_k^i(n)$$

In addition,

$$-h(n)x_{n-1}' = -h(n)x_m'(t_{n-1}) = \sum_{i=0}^{m} i\, d_i$$

$$\vdots$$

$$-h(n)x_{n-s}' = -h(n)x_m'(t_{n-s}) = \sum_{i=0}^{m} i\, d_i \tau_s^{i-1}(n)$$

Our equations for the coefficients now become

$$\begin{bmatrix} 1 & 0 & 0 & \cdots\cdots & 0 \\ 1 & 1 & 1 & \cdots\cdots & 1 \\ 1 & \tau_2(n) & \tau_2^2(n) & \cdots\cdots & \tau_2^m(n) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \tau_k(n) & \tau_k^2(n) & \cdots\cdots & \tau_k^2(n) \\ 0 & 1 & 2 & \cdots\cdots & m \\ 0 & 1 & 2\cdot 2 & \cdots\cdots & m\,2^{m-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & 2\,\tau_s(n) & \cdots\cdots & m\,\tau_s^{m-1}(n) \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_k \\ d_{k+1} \\ \vdots \\ \vdots \\ d_m \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ \vdots \\ x_{n-k} \\ -h\,x'_{n-1} \\ \vdots \\ \vdots \\ -h\,x'_{n-s} \end{bmatrix}$$

We can rewrite this as

$$V^T(n)\,\varphi_p^T = e_1 \quad \Rightarrow \quad \varphi_p^T \equiv \varphi_p^T(n)$$

which indicates that our discretization formulas may be *different* in each step. In other words, the general approximation will now have the form

$$-h(n)x'_n \equiv d_1 = \sum_{j=0}^{k} a_j(n)\,x_{n-j} - h(n)\sum_{j=1}^{s} b_j(n)\,x'_{n-j}$$

implying that coefficients $a_j(n)$ and $b_j(n)$ need to be recomputed every time the step changes

COMMENT. This process seems very inefficient. However, we should point out that for lower order methods the formulas are still *independent of h*. For example, the equations for the trapezoidal method are

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n-1} \\ -hx_{n-1} \end{bmatrix}$$

so in that case $V \neq V(n)$. As a result, the coefficients a_0, a_1 and b_1 are indeed independent of n.

Local Truncation Error

Based on what we established so far, it makes sense to use the *trapezoidal method* in conjunction with a *variable step*, $h(n)$. How can we select an appropriate value for $h(n)$? To see this, we need to examine the accuracy of the trapezoidal approximation.

To evaluate the error in computing x_n, we will assume that *all* the previously computed points are perfectly accurate, (*i. e.* $x_n = x(t_n)$, ..., $x_{n-k} = x(t_{n-k})$), and consider only the error incurred in this step. For simpler notation, in the following we will use h instead of $h(n)$.

From Taylor's formula, we have

$$x(t_n) = x(t_{n-1}) + h\,x'(t_{n-1}) + \frac{h^2}{2}x''(t_{n-1}) + \frac{h^3}{6}x'''(t_{n-1}) + \;.\;.\;.$$

Similarly, setting $y(t_n) \equiv x'(t_n)$ we can write

$$y(t_n) = y(t_{n-1}) + h\,y'(t_{n-1}) + \frac{h^2}{2}y''(t_{n-1}) + \frac{h^3}{6}y'''(t_{n-1}) + \;.\;.\;.$$

which implies that

$$x'(t_n) = x'(t_{n-1}) + h\,x''(t_{n-1}) + \frac{h^2}{2}x'''(t_{n-1}) + \frac{h^3}{6}x^{(4)}(t_{n-1}) + \;.\;.\;.$$

This last formula allows us to express the second derivative as

$$x''(t_{n-1}) = \frac{1}{h}x'(t_n) - \frac{1}{h}x'(t_{n-1}) - \frac{h}{2}x'''(t_{n-1}) - \frac{h^2}{6}x^{(4)}(t_{n-1}) +$$

Substituting this back into the original Taylor series expansion, we obtain the following expression for $x(t_n)$

$$x(t_n) = x(t_{n-1}) + \frac{h}{2}x'(t_n) + \frac{h}{2}x'(t_{n-1}) - \frac{h^3}{12}x'''(t_{n-1}) - \;.\;.\;.$$

The first three terms on the right hand side correspond to the *trapezoidal formula*, and the remainder represents *the local truncation error* (that is, the error of the trapezoidal approximation in a *single* step). Given that h is small, this error can be estimated as

$$E_n \approx \frac{h^3(n)}{12} x_{n-1}'''$$

In performing a transient analysis, we are typically given an error bound ε, defined as

$$\varepsilon \equiv \frac{total\ permissible\ error\ at\ t_{end}}{t_{end}}$$

In order to satisfy this specification, in any given step we can allow only a *fraction* of the total error. In other words, E_n can not exceed

$$E_n = \frac{h(n)}{t_{end}} \cdot \left[total\ permissible\ error\ at\ t_{end}\right] \equiv h(n) \cdot \varepsilon$$

Using the expression for E_n, we now have

$$\frac{h^3(n)}{12} x_{n-1}''' = h(n) \cdot \varepsilon$$

which allows us to calculate $h(n)$ as

$$h(n) = \sqrt{\frac{12\varepsilon}{x'''_{n-1}}}$$

This is precisely how time step control is implemented in SPICE (note that the third derivative at time t_{n-1} can be computed easily using divided differences).

ADVANCED DEVICE MODELS

In our previous analysis of MOSFETS, we assumed that dynamic behavior of these devices can be modeled by two *constant* capacitances, C_{gs} and C_{gd}. In the case of diodes and bipolar transisors, our models ignored capacitive effects altogether. At high frequencies, however, capacitive effects become very important, and must be studied in more detail. In other words, we will not only have to include additional capacitors in our models, but will also have to treat them as *nonlinear*.

Nonlinear capacitors

For a *linear* capacitor, the charge and voltage are related as $q = CV$. However, for semiconducor devices, this simple relationship is no longer true.

EXAMPLE 1. The capacitance of a *pn* junction appears in models of diodes and bipolar transistors.

a) When the junction is *reversely biased*, the charge and voltage are related as

$$q = K_1 \left[1 - \left(1 - \frac{V_j}{\Phi} \right)^{1-M} \right]$$

where V_j represents the voltage across the junction, and K_1, Φ and M are junction parameters.

b) When the junction is *forward biased*, the charge and voltage are related as

$$q = K_2 \left(e^{\frac{V_j}{V_T}} - 1 \right)$$

Obviously, in both cases, the q - V relationship is *nonlinear*.

EXAMPLE 2. In MOSFET devices there are separate charges associated with the drain, gate, source and bulk. The q - V relationships are very complicated, and depend on the region of operation. We will show these relationships for the *linear region only* (where $V_{DS} < V_{GS} - V_T$).

a) *Gate charge*:

$$Q_G = K_1 \left[V_{GS} - K_2 - \frac{1}{2} V_{DS} + \frac{1}{12} \frac{\alpha V_{DS}^2}{\left(V_{GS} - V_T - \frac{\alpha}{2} V_{DS} \right)} \right]$$

b) *Bulk charge*:

$$Q_B = K_1 \left[K_3 - \frac{(1-\alpha)}{2} V_{DS} - \frac{1}{12} \frac{(1-\alpha)\alpha V_{DS}^2}{\left(V_{GS} - V_T - \frac{\alpha}{2} V_{DS}\right)} \right]$$

c) *Drain charge*:

$$Q_D = -K_1 \left[\frac{1}{2}(V_{GS} - V_T) - \frac{3}{4} \alpha V_{DS} + \frac{1}{8} \frac{\alpha^2 V_{DS}^2}{\left(V_{GS} - V_T - \frac{\alpha}{2} V_{DS}\right)} \right]$$

d) *Source charge*:

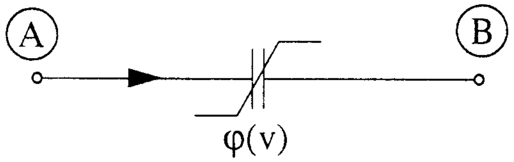
$$Q_S = -K_1 \left[\frac{1}{2}(V_{GS} - V_T) + \frac{1}{4} \alpha V_{DS} - \frac{1}{24} \frac{\alpha^2 V_{DS}^2}{\left(V_{GS} - V_T - \frac{\alpha}{2} V_{DS}\right)} \right]$$

In these formulas, $\alpha \equiv a + b(V_{GS} - V_T)$, where *a* and *b* are *short channel parameters*. K_1, K_2, and K_3 are *constants,* which depend on the properties of the MOSFET (width, length, oxide thickness, etc.), and V_T additionally depends on voltage V_B (the so called *body effect*).

Apart from being complicated, these q - V relationships create an additional problem. Namely, each of the charges depends on *four different voltages* : $Q_i = f_i\ (V_G, V_S, V_D, V_B)$. As a result, we need to introduce the concept of *distributed capacitances*, where a 4 × 4 *capacitance matrix* is associated with each device.

Circuit Analysis With Nonlinear Capacitors

To keep the analysis relatively simple, in the following we will disregard distributed capacitances, and consider only q - V relationships of the form $q = \varphi(V)$. The symbol for a nonlinear capacitor and its contribution to circuit equations are shown below.



$A \ldots \ +q'$

$B \ldots \ -q'$

$COM\!: \quad q - \varphi(v) = 0$

The corresponding stamp contributes to E, G and $p(x)$, and introduces *charge* q as a new variable.

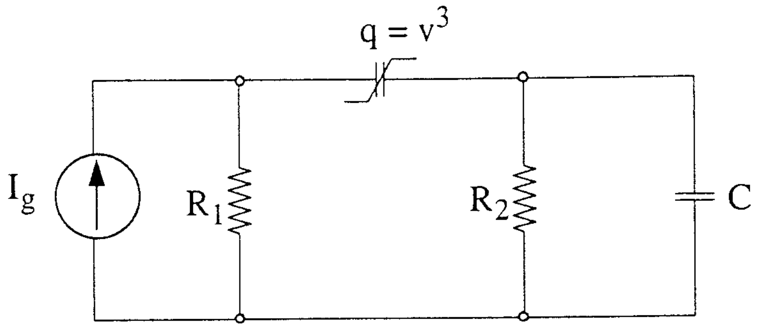
Contribution to E

$$\begin{array}{c} \\ A \\ B \\ COM \end{array}\begin{array}{c} \begin{array}{ccc} V_A' & V_B' & q' \end{array} \\ \left[\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ \left[\begin{array}{c} V_A' \\ V_B' \\ q' \end{array}\right] \end{array}$$

Contribution to G and $p(x)$

$$\begin{array}{c|ccc} A & 0 & 0 & 0 \\ B & 0 & 0 & 0 \\ COM & 0 & 0 & 1 \end{array} \quad \left[\begin{array}{c} V_A \\ V_B \\ q \end{array}\right] \quad ; \quad \begin{array}{c|c} A & 0 \\ B & 0 \\ COM & -\varphi(V_A - V_B) \end{array}$$

EXAMPLE



Stamps for resistors (contribution to G only):

$$R_1: \quad \begin{array}{cc} & V_1 \\ 1 & \left[\dfrac{1}{R_1} \right] \end{array} ; \qquad R_2: \quad \begin{array}{cc} & V_2 \\ 2 & \left[\dfrac{1}{R_2} \right] \end{array}$$

Stamp for current source

$$I_g: \quad 1 \left[I_g \right]$$

Stamp for linear capacitor

$$C: \quad \begin{array}{cc} & V_2' \\ 2 & \left[C \right] \end{array}$$

Stamp for nonlinear capacitor $q = v^3$

$$q: \begin{array}{c|ccc|} & V_1' & V_2' & q' \\ 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & -1 \\ COM & 0 & 0 & 0 \end{array} +$$

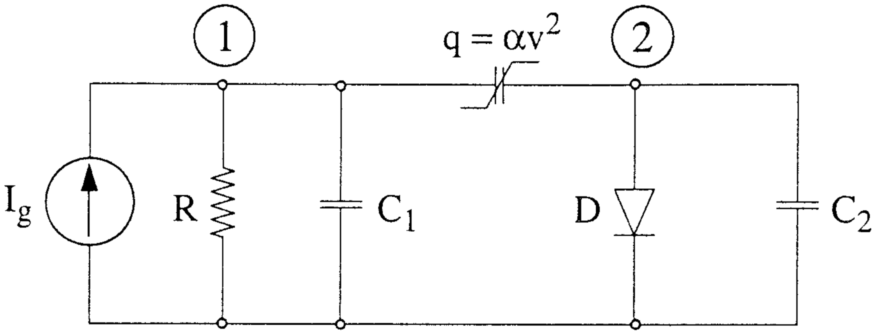
$$+ \begin{array}{c|ccc|} & V_1 & V_2 & q \\ 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & -1 \\ COM & -1 & 1 & 0 \end{array} + \begin{array}{c|c|} 1 & 0 \\ 2 & 0 \\ COM & -(V_1 - V_2)^3 \end{array}$$

Combining all the stamps, we obtain

$$\begin{array}{c} \begin{array}{ccc} V_1' & V_2' & q' \end{array} \\ \left[\begin{array}{ccc} 0 & 0 & 1 \\ 0 & C & -1 \\ 0 & 0 & 0 \end{array}\right] \end{array} \begin{bmatrix} V_1' \\ V_2' \\ q' \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -(V_1 - V_2)^3 \end{bmatrix} +$$

$$+ \begin{array}{c} \begin{array}{ccc} V_1 & V_2 & q \end{array} \\ \left[\begin{array}{ccc} \frac{1}{R_1} & 0 & 0 \\ 0 & \frac{1}{R_2} & 0 \\ 0 & 0 & 1 \end{array}\right] \end{array} \begin{bmatrix} V_1 \\ V_2 \\ q \end{bmatrix} - \begin{bmatrix} I_g \\ 0 \\ 0 \end{bmatrix} = 0$$

EXAMPLE



Stamp for resistor (contribution to G only):

$$R: \quad \begin{array}{cc} & V_1 \\ 1 & \left[\dfrac{1}{R} \right] \end{array}$$

Stamp for current source

$$I_g: \quad 1 \left[I_g \right]$$

Stamp for linear capacitors

$$C_1: \quad \begin{array}{cc} & V_1' \\ 1 & \left[C_1 \right] \end{array} \quad ; \quad C_2: \quad \begin{array}{cc} & V_2' \\ 2 & \left[C_2 \right] \end{array}$$

Stamp for nonlinear resistor

$$D: \quad 2\left[I_S\left(e^{\frac{V_2}{V_T}} - 1\right)\right]$$

Stamp for nonlinear capacitor

$$q: \quad \begin{array}{c|ccc|} & V_1' & V_2' & q' \\ A & 0 & 0 & 1 \\ B & 0 & 0 & -1 \\ COM & 0 & 0 & 0 \end{array} \quad ;$$

$$\begin{array}{c|ccc|} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ COM & 0 & 0 & 1 \end{array} \quad ; \quad \begin{array}{c|c|} 1 & 0 \\ 2 & 0 \\ COM & -\alpha(V_1 - V_2)^2 \end{array}$$

Combining all the stamps

$$\begin{bmatrix} C_1 & 0 & 1 \\ 0 & C_2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1' \\ V_2' \\ q' \end{bmatrix} + \begin{bmatrix} \frac{1}{R} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ q \end{bmatrix} +$$

$$+ \begin{bmatrix} 0 \\ I_S\left(e^{\frac{V_2}{V_T}} - 1\right) \\ -\alpha(V_1 - V_2)^2 \end{bmatrix} - \begin{bmatrix} I_g \\ 0 \\ 0 \end{bmatrix} = 0$$

Transient Analysis With Nonlinear Capacitors

Given that nonlinear capacitors introduce charges as additional variables and a set of algebraic relationships between q and V, the circuit equations will have the form

$$\begin{bmatrix} E_{11} & E_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V' \\ q' \end{bmatrix} + \begin{bmatrix} G_{11} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} V \\ q \end{bmatrix} + \begin{bmatrix} p_1(V) \\ p_2(V) \end{bmatrix} + \begin{bmatrix} w_1 \\ 0 \end{bmatrix} = 0$$

We can break this up into *two* separate sets of equations

$$E_{11} V' + E_{12} q' + G_{11} V + p_1(V) - w_1 = 0$$

$$q - p_2(V) = 0$$

Using the *backward Euler method*, the first set of equations can be approximated at $t = t_n$ as

$$\frac{1}{h} E_{11} [V_n - V_{n-1}] + \frac{1}{h} E_{12} [q_n - q_{n-1}] + G_{11} V_n + p_1(V_n) - w_1(t_n) = 0$$

Since $q_n = p_2(V_n)$ is known, this will be a *nonlinear algebraic equation*

$$F(V_n) \equiv \frac{1}{h} E_{11} [V_n - V_{n-1}] + \frac{1}{h} E_{12} [p_2(V_n) - p_2(V_{n-1})] +$$

$$+ G_{11} V_n + p_1(V_n) - w_1(t_n) = 0$$

Setting $x \equiv V_n$ and $y \equiv V_{n\text{-}1}$ (as we did before in transient simulations), it follows that in each time point we need to solve

$$\left[\frac{1}{h}E_{11} + G_{11}\right]x + \frac{1}{h}E_{12}\,p_2(x) + p_1(x) -$$

$$-\left[\frac{1}{h}E_{11}\,y + \frac{1}{h}E_{12}\,p_2(y) + w_1(t_n)\right] = 0$$

This equation can be solved by Newton's method. The Jacobian will be

$$J(x) = \frac{1}{h}E_{11} + G_{11} + \frac{1}{h}E_{12}\frac{\partial p_2(x)}{\partial x} + \frac{\partial p_1(x)}{\partial x}$$

Note that

$$J_{DC}(x) \equiv G_{11} + \frac{\partial p_1(x)}{\partial x}$$

corresponds to the Jacobian used in DC analysis (for which we already know how to construct the stamps).