Electric Circuits I

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Project 2: Basic Filter Design

The circuit below is driven by a sinusoidal voltage source $V_g(t) = \cos\omega t$.

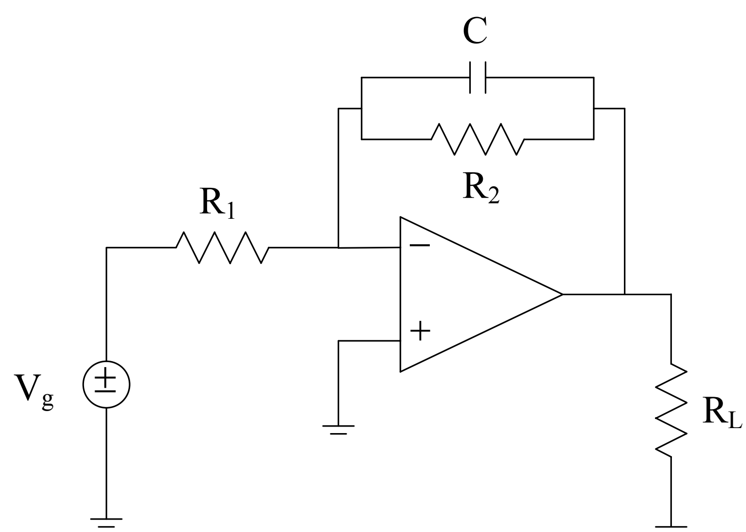

Fig. 1. A simple low-pass filter.

Problem 1. Derive an expression for the amplitude of the output voltage as a function of R_1, R_2, C and ω (in the following we will refer to this amplitude as $A(\omega)$).

Problem 2. Based on the results of Problem 1, design a low-pass filter that satisfies the following requirements:

1) For $\omega = 0$ (which corresponds to DC), the amplitude must be $A(0) = 1$.
2) For $\omega = 1{,}000\, rad/s$, the amplitude must be $A(1{,}000) = 0.7$.
3) Your element values must be physically realistic.

Problem 3. Write an m-file that solves the circuit in Fig. 1 for different frequencies. Use it to plot $20\log A(\omega)$ for the values chosen in Problem 2, and verify that the design requirements are satisfied.

Problem 4. Repeat Problem 1 for the circuit shown in Fig. 2.

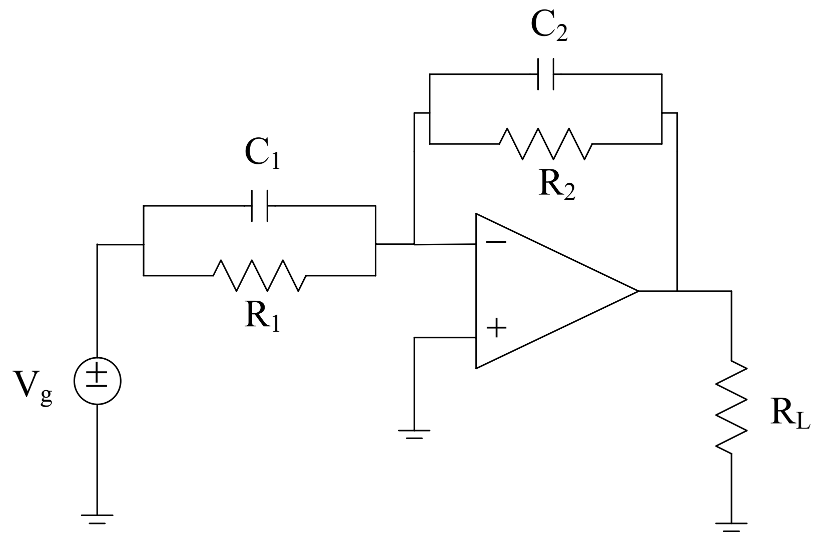

Fig. 2. An alternative circuit for filter design.

Problem 5. Choose physically realistic values for R_1, R_2, C_1 and C_2 so that $A(\omega) = 10$ for *all* values of ω (this would correspond to an *all-pass* filter).

Problem 6. Choose physically realistic values for R_1, R_2, C_1 and C_2 so that the following two requirements are met:

1) For $\omega = 0$ the amplitude must be $A(0) = 0.01$.

2) For $\omega = 10,000\, rad/s$, the amplitude must be $A(10,000) = 0.9$.

What kind of filter is this? Explain.

Problem 7. Write an m-file that solves the circuit in Fig. 2 for different frequencies, and plot $20 \log A(\omega)$ for the values chosen in Problems 5 and 6. Use these plots to verify the two designs.

Problem 8. Assemble the circuits in Figs. 1 and 2 with the element values obtained in Problems 2, 5 and 6. In all three cases measure $A(\omega)$ for a range of relevant frequencies, and use the data to plot $20 \log A(\omega)$ (do this in Matlab). Compare the measurements with your simulation results.

Project 2: Analysis of Basic Filters

In this section our objective will be to study basic filters and their applications. We begin with the following simple circuit, in which the source has the form $v_g(t) = \cos \omega t$.

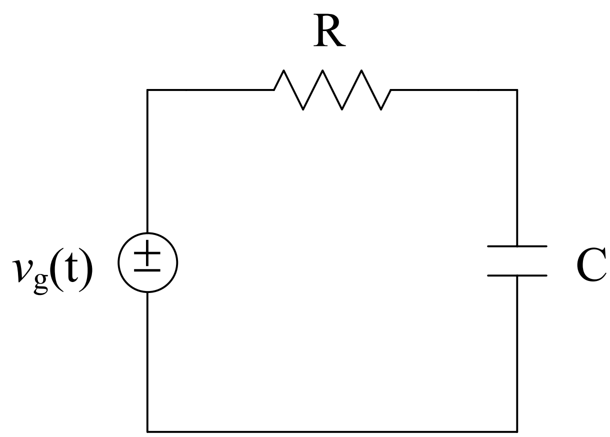

Fig. 5. A simple RC filter.

Since we would like to use this circuit at different frequencies, it makes sense to leave ω unspecified in our analysis. Taking the capacitor voltage as the output, we have

$$\vec{V}_0 = \frac{1/(j\omega C)}{R + 1/(j\omega C)} \vec{V}_g = \frac{1}{1 + j\omega RC} \tag{17}$$

(since $\vec{V}_g = 1$), and the magnitude and angle of $\vec{V}_0$ can be expressed as

$$|\vec{V}_0| \equiv A(\omega) = \frac{1}{\sqrt{1 + (\omega RC)^2}} \tag{18}$$

and

$$\angle \vec{V}_0 \equiv \varphi(\omega) = -\tan^{-1}(\omega RC) \tag{19}$$

respectively. The corresponding output voltage in the time domain will have the form

$$v_0(t) = A(\omega)\cos(\omega t + \varphi(\omega)) \tag{20}$$

It is important to recognize that *both* the amplitude *and* the angle of the output voltage (20) depend on the frequency. To see what this means in practical terms, let us focus on three specific values of ω :

1) For $\omega = 0$ (which corresponds to DC), we have $A(0) = 1$ and $\varphi(0) = 0$, and consequently

$$v_0(t) = 1 \tag{21}$$

2) For $\omega = \omega_0 \equiv 1/RC$, we have $A(\omega_0) = 1/\sqrt{2} = 0.71$ and $\varphi(\omega_0) = -45°$, and consequently

$$v_0(t) = 0.71\cos(\omega_0 t - 45°) \tag{22}$$

3) For $\omega = 20\omega_0$, we have $A(20\omega_0) = 0.05$ and $\varphi(20\omega_0) = -87°$, and consequently

$$v_0(t) = 0.05\cos(20\omega_0 t - 87°) \tag{23}$$

Since the amplitude of the output voltage becomes very small when $\omega \gg \omega_0$, we refer to this type of circuit as a *low-pass filter*. The frequency ω_0 at which the amplitude of $v_0(t)$ is reduced to $\approx 70\%$ of its initial value is an important characteristic of the filter, and is known as the *corner frequency*.

For a more detailed analysis of this circuit, it is useful to obtain a plot of the function $A(\omega)$ given in (18). Such a plot is shown in Fig. 6, for the case when $R = 1K\Omega$ and $C = 1\mu F$ (the Matlab program that was used to obtain it is provided in in Appendix 2).

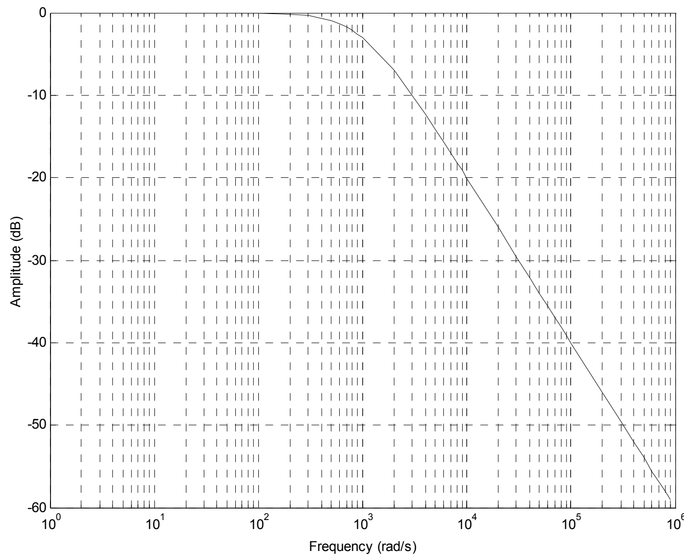

Fig. 6. Frequency response for the circuit in Fig. 5.

Several comments need to be made regarding this plot.

Remark 1. It is customary to plot $20 \log A(\omega)$ rather than $A(\omega)$ itself.

Remark 2. The x-axis is logarithmic, since the frequency ranges from $1\,rad/s$ to $10^6\,rad/s$.

Remark 3. The corner frequency for this particular low-pass filter is $\omega_0 = 1/RC = 1,000\,rad/s$.

A Typical Application of Filters

Suppose you are receiving a signal which has an undesirable high frequency component (we refer to such a component as "interference"). This type of situation occurs virtually every time information is transmitted over a distance. A typical example of such a signal would be something like

$$v_{in}(t) = \cos 10t + 0.2 \cos 300t \tag{24}$$

which corresponds to the function shown in Fig. 7.

The simple low-pass circuit in Fig. 5 can be used to "clean up" such a signal. To see this, let us pick R and C so that $\omega_0 = 1/RC = 20\,rad/s$, and use superposition to separate the responses. For the two components, the expressions derived in (18) and (19) produce

$$A(10) = 0.894 \quad \text{and} \quad \varphi(10) = -26^\circ \tag{25}$$

and

$$A(300) = 0.013 \quad \text{and} \quad \varphi(300) = -86^\circ, \tag{26}$$

respectively. Combining the two responses, we obtain

$$v_0(t) = 0.894 \cos(10t - 26^\circ) + 0.013 \cos(300t - 86^\circ) \tag{27}$$

which is graphically represented in Fig. 8.

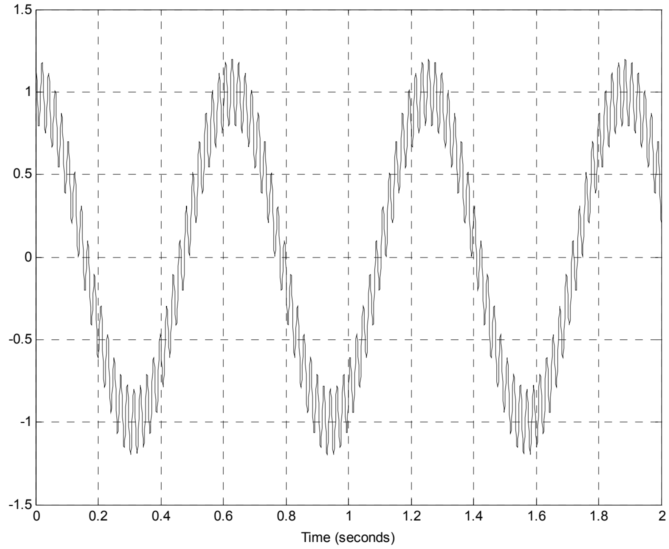

Fig. 7. A noisy sinusoid.

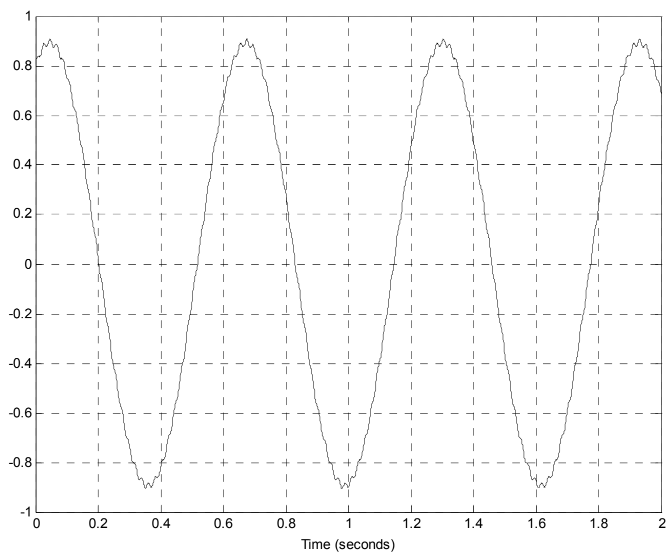

Fig. 8. The sinusoid after filtering.

This plot clearly shows that the undesirable component has now been almost eliminated (at the expense of a 10% decrease in the amplitude of the output voltage).

The Design and Simulation of Active Filters

Although the circuit considered in Fig. 5 is a legitimate low-pass filter, it can satisfy only very basic design requirements. For more sophisticated applications, it is necessary to use so-called *active circuits*, with one or more operational amplifiers. A class of simple active filters corresponds to the circuit in Fig. 9.

This generic circuit can produce several different types of filters, depending on how we choose impedances Z_1 and Z_2. To analyze it, let us first observe that the KCL equation at the inverting terminal of the op amp has the form

$$-\vec{I}_{Z1} + \vec{I}_{Z2} = 0 \tag{28}$$

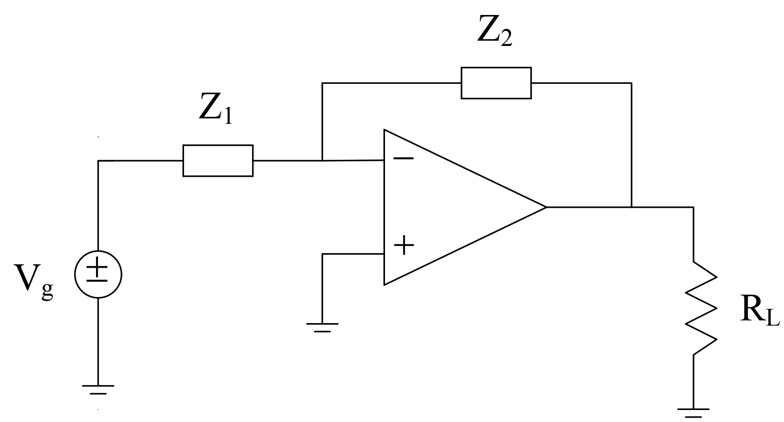

Fig. 9. A generic active filter.

Recalling that this input is virtually grounded, we have

$$\vec{I}_{Z1} = \frac{\vec{V}_g}{Z_1} \quad \text{and} \quad \vec{I}_{Z2} = -\frac{\vec{V}_0}{Z_2} \tag{29}$$

and therefore

$$\vec{V}_0 = -\frac{Z_2}{Z_1}\,\vec{V}_g. \tag{30}$$

As an illustration of what this circuit can do, let us consider the scenario shown in Fig. 10, which corresponds to $Z_1 = R_1 + 1/j\omega C$ and $Z_2 = R_2$.

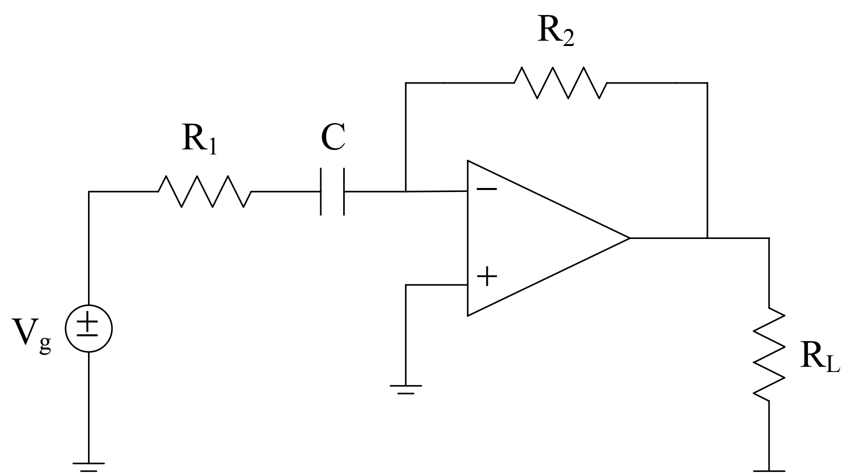

Fig. 10. A special case of the active filter circuit.

In this case (30) becomes

$$\vec{V}_0 = -\frac{R_2}{R_1 + 1/j\omega C}\,\vec{V}_g = -\frac{j\omega C R_2}{1 + j\omega C R_1} \tag{31}$$

and the amplitude of the output voltage is obtained as

$$A(\omega) = \frac{\omega C R_2}{\sqrt{1 + (\omega C R_1)^2}} \tag{32}$$

This function is shown in Fig. 11, for the case when $R_1 = R_2 = 1K\Omega$ and $C = 1\mu F$ (the Matlab program used to obtain it is provided in Appendix 3).

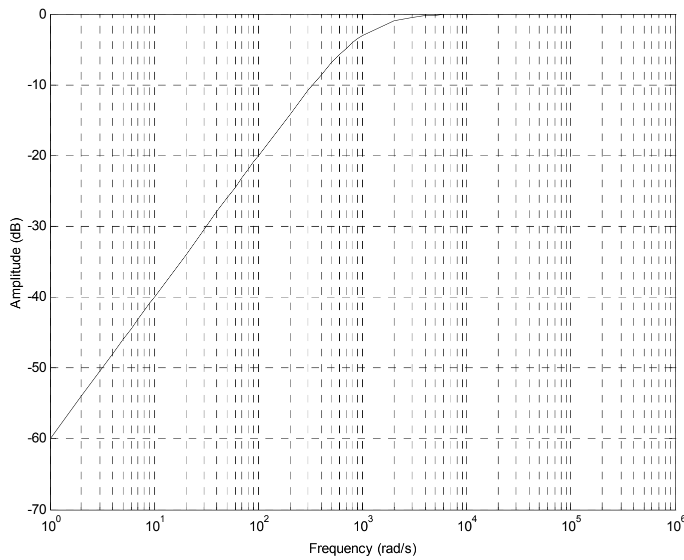

Fig. 11. The frequency response of the circuit in Fig. 10.

The plot suggests that the circuit in Fig. 10 is a *high-pass filter*, which produces meaningful output voltages for sufficiently high frequencies.

For larger circuits, it is generally very difficult to obtain an explicit expression for $A(\omega)$ like the one in (32), and it is necessary to resort to simulation. To illustrate how this process works, let us once again consider the circuit in Fig. 10. The KCL equations for this circuit have the form

$$\begin{array}{lcl} & & \\ 1)\ \ \vec{I}_g + \vec{I}_{R1} = 0 & \Rightarrow & \vec{V}_1 = \vec{V}_g \\ 2)\ \ -\vec{I}_{R1} + \vec{I}_C = 0 & & \\ 3)\ \ -\vec{I}_C + \vec{I}_{R2} = 0 & & \\ 4)\ \ -\vec{I}_{R1} + \vec{I}_x + \vec{I}_{RL} = 0 & \Rightarrow & \vec{V}_3 = 0 \end{array} \qquad \begin{array}{l} \vec{I}_{R1} = (\vec{V}_1 - \vec{V}_2)/R_1 \\ \vec{I}_{R2} = (\vec{V}_3 - \vec{V}_4)/R_2 \\ \vec{I}_{RL} = \vec{V}_4/R_L \\ \vec{I}_C = j\omega C(\vec{V}_2 - \vec{V}_3) \\ \vec{I}_g = ? \\ \vec{I}_x = ? \end{array} \tag{33}$$

It is not difficult to see that these equations can be rewritten in matrix form as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/R_1 & (1/R_1 + j\omega C) & -j\omega C & 0 \\ 0 & -j\omega C & (1/R_2 + j\omega C) & -1/R_2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \vec{V}_1 \\ \vec{V}_2 \\ \vec{V}_3 \\ \vec{V}_4 \end{bmatrix} = \begin{bmatrix} \vec{V}_g \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{34}$$

Since ω is a variable, it is useful to further rewrite (34) as

$$(G_1 + j\omega G_2)\vec{V} = b \tag{35}$$

where

$$G_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/R_1 & 1/R_1 & 0 & 0 \\ 0 & 0 & 1/R_2 & -1/R_2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{36}$$

and

$$G_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & C & -C & 0 \\ 0 & -C & C & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{37}$$

are *fixed* matrices, $\vec{V}$ is the vector of unknown voltages, and

$$b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{38}$$

(note again that $\vec{V}_g = 1$ for all problems of this type).

The Matlab function in Appendix 4 can be used to simulate the frequency response of this circuit, based on repeated solutions of equation (35) for different values of ω. The frequency response obtained in this way is identical to the one shown in Fig. 11.

Appendix 2

```
function F=freqresp1(R,C,w)

p1=(w.^2)*(R^2)*(C^2);
% The command w.^2 squares each element of w separately.

p2=ones(1,length(w));
% p2 is a row of ones, whose size matches w.

den=sqrt(p1+p2);
% Each element of den represents the denominator at a
% different frequency.

num=ones(1,length(w));
% The numerator is independent of the frequency, and is
% always equal to 1.

A=num./den;
% The command ./ divides each element of vector num by the
% corresponding element of vector den. As a result, the elements
% of A represent the amplitude at different frequencies.

F=20*log10(A);
% This is necessary in order to represent the amplitude in decibels.
```

Appendix 3

```
function F=freqresp2(R1,R2,C,w)

p1=(w.^2)*(R1^2)*(C^2);
% The command w.^2 squares each element of w separately.

p2=ones(1,length(w));
% p2 is a row of ones, whose size matches w.

den=sqrt(p1+p2);
% Each element of den represents the denominator at a
% different frequency.

num=w*C*R2;
% In this case, the numerator is a function of the frequency.

A=num./den;
% The command ./ divides each element of vector num by the
% corresponding element of vector den. As a result, the elements
% of A represent the amplitude at different frequencies.

F=20*log10(A);
% This is necessary in order to represent the amplitude in decibels.
```

Appendix 4

```
function F=freqresp3(G,C,b,w)

mag=zeros(4,1);
% We need to initialize vector mag, in order not
% to confuse Matlab.

for k=1:length(w)
   omega=w(k);
   % Omega is the next frequency.

   A=G+i*omega*C;
   x=A\b;
   % Vector x contains the solution of the node voltage
   % equations in complex form.

   mag=[mag abs(x)];
   % The magnitudes of all voltages as stored as a new column in
   % matrix mag.
end

% Each column of matrix mag contains voltage magnitudes computed at
% a particular frequency. Note, however, that the first column of
% this matrix is essentially redundant. It is a vector of zeros that
% serves only as a place holder.

V4=mag(4,2:length(w)+1);
% Since we are interested only in V4, we will extract row 4 of matrix
% mag (and ignore the first column).

F=20*log10(V4);
```