ELEN 236

HANDOUT #6

1. Given dynamic system

$$S: \quad \begin{matrix} \dot{x}=Ax+Bu \\ y=Cx \end{matrix}$$

where:

$$A=\begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ -\frac{2}{5} & -\frac{11}{5} \end{bmatrix} \quad ; \quad B=\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad ; \quad C=[1 \quad 0]$$

a) Compute the eigenvalues of A and the corresponding eigenvectors.
b) Based on part a), find the transformation that results in the Jordan canonical form for A.
c) Apply this transformation to system S and see if the resulting differential equations are decoupled.

2. Repeat problem 1. in case:

$$A=\begin{bmatrix} -\frac{5}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{2} & \frac{1}{2} \\ 0 & 0 & -2 \end{bmatrix} \quad ; \quad B=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad ; \quad C=[1 \quad 0 \quad 0]$$

SOLUTIONS

1. Given

$$A=\begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ -\frac{2}{5} & -\frac{11}{5} \end{bmatrix}$$

we first need to find its eigenvalues. We have:

$$A-\lambda I=\begin{bmatrix} -\left(\lambda+\frac{4}{5}\right) & \frac{3}{5} \\ -\frac{2}{5} & -\left(\lambda+\frac{11}{5}\right) \end{bmatrix}$$

and therefore

$$|A-\lambda I|=\lambda^2+3\lambda+2 \quad \Rightarrow \quad \lambda_1=-1 \quad ; \quad \lambda_2=-2$$

Next, we compute the corresponding eigenvectors:

i) $\boldsymbol{\lambda_1=-1}$ We have:

$$\begin{bmatrix} \frac{1}{5} & \frac{3}{5} \\ -\frac{2}{5} & -\frac{6}{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Multiplying row 1 by 2 and adding it to row 2, we have

$$\begin{bmatrix} \frac{1}{5} & \frac{3}{5} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Setting $x_2=t \Rightarrow x_1=-3t$, so:

$$\underline{m_1} = \begin{bmatrix} -3t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

There are infinitely many choices for t, and for convenience we usually take t=1. $\Rightarrow$

$$\underline{m_1} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

ii) λ_2**=-2** We have:

$$\begin{bmatrix} \frac{6}{5} & \frac{3}{5} \\ -\frac{2}{5} & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Multiplying row 1 by 1/3 and adding it to row 2, we get

$$\begin{bmatrix} \frac{6}{5} & \frac{3}{5} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Setting $x_2=t \Rightarrow x_1=-1/2t$. Now,

$$\underline{m_2} = \begin{bmatrix} -\frac{1}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Choosing t=1 as before, we obtain:

$$\underline{m_2} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Having found eigenvectors m_1 and m_2, we have actually computed the transformation matrix M:

$$M = \begin{bmatrix} \underline{m_1} & \underline{m_2} \end{bmatrix} = \begin{bmatrix} -3 & -\frac{1}{2} \\ 1 & 1 \end{bmatrix}$$

It is now easily verified that

$$M^{-1} = -\frac{2}{5} \times \begin{bmatrix} 1 & \frac{1}{2} \\ -1 & -3 \end{bmatrix}$$

and that

$$M^{-1}AM = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

which is the Jordan canonical form. We can now apply transformation $z=M^{-1}x$ to our original system, obtaining:

$$\dot{z}=M^{-1}AMz+M^{-1}Bu$$

$$y=CMz$$

Since

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad C = [1 \quad 0]$$

this results in:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} . \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} -\frac{1}{5} \\ \frac{6}{5} \end{bmatrix} u$$

$$y = \begin{bmatrix} -\frac{2}{5} & -\frac{1}{5} \end{bmatrix} z$$

The transformed system is now in the desired Jordan canonical form, and the differential equations are indeed decoupled.

2) Here we have the same type of problem as the previous one, only a little more tedious for computation. We begin again by forming

$$A-\lambda I = \begin{bmatrix} -\left(\lambda+\frac{5}{2}\right) & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\left(\lambda+\frac{5}{2}\right) & \frac{1}{2} \\ 0 & 0 & -(\lambda+2) \end{bmatrix}$$

The characteristic polynomial is:

$$|A-\lambda I| = \lambda^3+7\lambda^2+16\lambda+12 = (\lambda+2)^2(\lambda+3)$$

Therefore, the eigenvalues are $\lambda_1=\lambda_2=-2$ and $\lambda_3=-3$. We now need to compute the eigenvectors.

i) **$\lambda=-2$** We have:

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Adding row 1 to row 2, this becomes

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Note that the second equation directly implies $x_3=0$. In view of this, we set $x_2=t \Rightarrow x_1=t$. Consequently,

$$\underline{m_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} t$$

Choosing t=1 $\Rightarrow$

$$\underline{m_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Observe now that λ=-2 is a *double* eigenvalue for which we found only *one* eigenvector. Therefore, here we will not have a purely diagonal Jordan form. We can compute $\underline{m_2}$ from equation:

$$\underline{A}\,\underline{m_2} = \underline{m_1} + \lambda \underline{m_2}$$

which can be rewritten explicitly as:

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Adding row 1 to row 2 we get:

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

The second equation implies $x_3=2$. Setting $x_2=t \Rightarrow x_1=t$, so we have:

$$\underline{m_2} = t\begin{bmatrix}1\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\2\end{bmatrix}$$

Choosing t=1, we obtain:

$$\underline{m_2} = \begin{bmatrix}1\\1\\2\end{bmatrix}$$

Finally, for $\lambda=-3$ we have:

$$\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0$$

Multiplying row 1 by -1 and adding it to row 2 we get:

$$\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0$$

Equation 3 implies $x_3=0$. Setting $x_2=t \Rightarrow x_1=-t$. Consequently

$$\underline{m_3} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} t$$

Again, for t=1

$$\underline{m_3} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Having obtained $\underline{m}_1$, $\underline{m}_2$ and $\underline{m}_3$, we directly have the transformation matrix

$$M = \begin{bmatrix} \underline{m_1} & \underline{m_2} & \underline{m_3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix}$$

Since

$$M^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

it is easily verified that

$$M^{-1}AM=\begin{bmatrix}-2 & 1 & 0\\ 0 & -2 & 0\\ 0 & 0 & -3\end{bmatrix}$$

Going back to the original dynamic system, we introduce transformation $z=M^{-1}x$ obtaining:

$$\dot{z}=M^{-1}AMz+M^{-1}Bu$$

$$y=CMz$$

For

$$B=\begin{bmatrix}0\\0\\1\end{bmatrix} \quad \text{and} \quad C=[1 \quad 0 \quad 0]$$

this results in:

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\end{bmatrix} = \begin{bmatrix}-2 & 1 & 0\\ 0 & -2 & 0\\ 0 & 0 & -3\end{bmatrix} \begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix} + \begin{bmatrix}-\frac{1}{2}\\ \frac{1}{2}\\ 0\end{bmatrix} u$$

$$y=\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & -\frac{1}{2}\end{bmatrix} z$$

This is the Jordan canonical form, but the first two equations are *not* completely decoupled.

Namely, we have:

$$\dot{z}_1 = -2z_1 + z_2 - \frac{1}{2}u$$

since the Jordan form is not purely diagonal.