ELEN 236

HANDOUT #4

1. Given system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x}$$

where u(t) is the step function and initial conditions are $x_1(0)=x_2(0)=1$. Compute the state vector $\underline{x}(t)$ and transfer function G(s) using the transition matrix.

2. Given multi-input, multi-output system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where $u_1(t)$ is the step function, $u_2(t)=\delta(t)$ and the initial conditions are $x_1(0)=0$; $x_2(0)=1$. Compute the state vector $\underline{x}(t)$ and the transfer function matrix $\underline{G}(s)$, using the transition matrix.

SOLUTIONS

1. We are given system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x}$$

with initial conditions $x_1(0) = x_2(0) = 1$. Since we need to solve it using the transition matrix, we form:

$$sI - A = \begin{bmatrix} s & -2 \\ 1 & s+3 \end{bmatrix}$$

Consequently,

$$(sI-A)^{-1} = \frac{1}{s^2+3s+2} \begin{bmatrix} s+3 & 2 \\ -1 & s \end{bmatrix}$$

Observing that $s^2+3s+2 = (s+1)(s+2)$, we have

$$\Phi(s) = \begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)} & \dfrac{2}{(s+1)(s+2)} \\ \\ \dfrac{-1}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix}$$

Before taking the inverse Laplace transform, we must perform a partial fraction expansion of each term. This yields:

$$\Phi(s) = \begin{bmatrix} \left(\frac{2}{s+1} - \frac{1}{s+2}\right) & \left(\frac{2}{s+1} - \frac{2}{s+2}\right) \\ \left(\frac{1}{s+2} - \frac{1}{s+1}\right) & \left(\frac{2}{s+2} - \frac{1}{s+1}\right) \end{bmatrix}$$

Now we can take the inverse Laplace transform, obtaining the transition matrix:

$$\Phi(t) = \begin{bmatrix} \left(2e^{-t} - e^{-2t}\right) & \left(2e^{-t} - 2e^{-2t}\right) \\ \left(e^{-2t} - e^{-t}\right) & \left(-e^{-t} + 2e^{-2t}\right) \end{bmatrix}$$

The solution can be written as:

$$x(t) = \Phi(t)\,x(0) + \int_0^t \Phi(t-\tau)\,Bu(\tau)\,d\tau$$

The first part is easily computed. Since $x_1(0)=x_2(0)=1$, we have:

$$\Phi(t)\,x(0) = \begin{bmatrix} 4e^{-t} - 3e^{-2t} \\ -2e^{-t} + 3e^{-2t} \end{bmatrix}$$

The second part is somewhat messier. First of all,

$$\Phi(t-\tau)\,B = \begin{bmatrix} 4e^{-(t-\tau)} - 3e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 3e^{-2(t-\tau)} \end{bmatrix}$$

Now, since $u(\tau)=1$, $(\tau \geq 0)$, the second term is computed as:

$$\int_0^t \left(4e^{-(t-\tau)}-3e^{-2(t-\tau)}\right)d\tau = \frac{5}{2}-4e^{-t}+\frac{3}{2}e^{-2t}$$

and

$$\int_0^t \left(-2e^{-(t-\tau)}+3e^{-2(t-\tau)}\right)d\tau = -\frac{1}{2}+2e^{-t}-\frac{3}{2}e^{-2t}$$

Therefore,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 4e^{-t}-3e^{-2t} \\ -2e^{-t}+3e^{-2t} \end{bmatrix} + \begin{bmatrix} \frac{5}{2}-4e^{-t}+\frac{3}{2}e^{-2t} \\ -\frac{1}{2}+2e^{-t}-\frac{3}{2}e-2t \end{bmatrix} \Longrightarrow$$

$$\Longrightarrow \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{5}{2}-\frac{3}{2}e^{-2t} \\ -\frac{1}{2}+\frac{3}{2}e^{-2t} \end{bmatrix}$$

It remains to compute G(s). This is easy, since we already know Φ(s). Recalling that

$$C = [1 \;\; 0] \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

we have

$$G(s) = C(sI-A)^{-1}B = \frac{s+5}{(s+1)(s+2)}$$

2. Here we have a multi-input, multi-output system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Again, since we are using the transition matrix approach, we first form:

$$(sI-A) = \begin{bmatrix} s & -1 \\ 3 & s+4 \end{bmatrix}$$

Then,

$$\Phi(s) = (sI-A)^{-1} = \frac{1}{(s+1)(s+3)} \begin{bmatrix} s+4 & 1 \\ -3 & s \end{bmatrix}$$

Applying partial fraction expansion to $\Phi(s)$, we have:

$$\Phi(s) = \begin{bmatrix} \left(\dfrac{\frac{3}{2}}{s+1} - \dfrac{\frac{1}{2}}{s+3} \right) & \left(\dfrac{\frac{1}{2}}{s+1} - \dfrac{\frac{1}{2}}{s+3} \right) \\ \left(\dfrac{-\frac{3}{2}}{s+1} + \dfrac{\frac{3}{2}}{s+3} \right) & \left(\dfrac{-\frac{1}{2}}{s+1} + \dfrac{\frac{3}{2}}{s+3} \right) \end{bmatrix}$$

Taking the inverse Laplace transform, we obtain the transition matrix as:

$$\Phi(t) = \begin{bmatrix} \left(\frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}\right) & \left(\frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t}\right) \\ \left(-\frac{3}{2}e^{-t} + \frac{3}{2}e^{-3t}\right) & \left(-\frac{1}{2}e^{-t} + \frac{3}{2}e^{-3t}\right) \end{bmatrix}$$

We are now ready to solve for $\underline{x}(t)$:

$$\underline{x}(t) = \Phi(t)\underline{x}(0) + \int_0^t \Phi(t-\tau) B\underline{u}(\tau)\,d\tau$$

Since $x_1(0)=0$, $x_2(0)=1 \Rightarrow$

$$\Phi(t)x(0) = \begin{bmatrix} \left(\frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t}\right) \\ \left(-\frac{1}{2}e^{-t} + \frac{3}{2}e^{-3t}\right) \end{bmatrix}$$

For computing the second term, we first obtain:

$$\Phi(t-\tau)B = \begin{bmatrix} \left(2e^{-t} - e^{-3t}\right) & -e^{-t} \\ \left(-2e^{-t} + 3e^{-3t}\right) & e^{-t} \end{bmatrix}$$

and

$$\Phi(t-\tau)B\underline{u}(\tau) = \begin{bmatrix} \left(2e^{-(t-\tau)} - e^{-3(t-\tau)}\right)u_1(\tau) - e^{-(t-\tau)}u_2(\tau) \\ \left(-2e^{-(t-\tau)} + 3e^{-3(t-\tau)}\right)u_1(\tau) + e^{-(t-\tau)}u_2(\tau) \end{bmatrix}$$

Recalling that $u_1(\tau)=1$ $(\tau\geq 0)$ and $u_2(\tau)=\delta(\tau)$, the two integrals are

$$\int_0^t \left(2e^{-(t-\tau)}-e^{-3(t-\tau)}\right)d\tau-\int_0^t e^{-(t-\tau)}\delta(\tau)\,d\tau=\frac{5}{3}-3e^{-t}+\frac{1}{3}e^{-3t}$$

and

$$\int_0^t \left(-2e^{-(t-\tau)}+3e^{-3(t-\tau)}\right)d\tau+\int_0^t e^{-(t-\tau)}\delta(\tau)\,d\tau=-1+3e^{-t}-e^{-3t}$$

Consequently

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}e^{-t}-\frac{1}{2}e^{-3t}\right) \\ \left(-\frac{1}{2}e^{-t}+\frac{3}{2}e^{-3t}\right) \end{bmatrix} + \begin{bmatrix} \left(\frac{5}{3}-3e^{-t}+\frac{1}{3}e^{-3t}\right) \\ \left(-1+3e^{-t}-e^{-3t}\right) \end{bmatrix} \Longrightarrow$$

$$\Longrightarrow \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{5}{3}-\frac{5}{2}e^{-t}-\frac{1}{6}e^{-3t}\right) \\ \left(-1+\frac{5}{2}e^{-t}+\frac{1}{2}e^{-3t}\right) \end{bmatrix}$$

To compute $\underline{G}(s)$, we have:

$$\underline{G}(s)=C(sI-A)^{-1}B$$

Having already computed $(sI-A)^{-1}$, we easily obtain

$$\underline{G}(s)=\begin{bmatrix} \frac{s+5}{(s+1)(s+3)} & -\frac{1}{s+1} \\ \frac{s-3}{(s+1)(s+3)} & \frac{1}{s+1} \end{bmatrix}$$

Note that $\underline{G}(s)$ is now a 2x2 matrix, since there are two inputs and two outputs.