HANDOUT #7

1. For matrix

$$A=\begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ -\frac{2}{5} & -\frac{11}{5} \end{bmatrix}$$

compute $f(A)=e^{At}$ using:

a) the Cayley-Hamilton theorem
b) Laplace transforms
c) the Jordan canonical form

2. Repeat parts a) and c) of problem 1. for $f(A)=A^K$.

3. For matrix

$$A=\begin{bmatrix} -1 & -\frac{5}{2} \\ 1 & 0 \end{bmatrix}$$

compute $f(A)=e^{At}$ using the Cayley-Hamilton theorem only.

SOLUTIONS

1. i) First we will use the approach based on the Cayley-Hamilton theorem. Matrix A is the one considered in the previous handout, where we showed that $\lambda_1=-1$ and $\lambda_2=-2$. Bearing in mind that A is a 2x2 matrix, we have:

$$f(x)=e^{xt} \quad \text{and} \quad R(x)=\alpha_o+\alpha_1 x$$

To compute α_o and α_1 we use:

$$e^{-t}\equiv f(\lambda_1)=R(\lambda_1)=\alpha_o+\alpha_1\lambda_1$$

$$e^{-2t}\equiv f(\lambda_2)=R(\lambda_2)=\alpha_o+\alpha_1\lambda_2$$

which gives rise to the following system of equations:

$$\begin{bmatrix}1 & -1\\ 1 & -2\end{bmatrix}\begin{bmatrix}\alpha_o\\ \alpha_1\end{bmatrix}=\begin{bmatrix}e^{-t}\\ e^{-2t}\end{bmatrix}$$

Solving this we easily obtain

$$\alpha_o=2e^{-t}-e^{-2t}$$

$$\alpha_1=e^{-t}-e^{-2t}$$

Consequently,

$$e^{At} = R(A) = \alpha_o I + \alpha_1 A = \begin{bmatrix} \left(\frac{6}{5}e^{-t} - \frac{1}{5}e^{-2t}\right) & \left(\frac{3}{5}e^{-t} - \frac{3}{5}e^{-2t}\right) \\ \left(-\frac{2}{5}e^{-t} + \frac{2}{5}e^{-2t}\right) & \left(-\frac{1}{5}e^{-t} + \frac{6}{5}e^{-2t}\right) \end{bmatrix}$$

ii) Using Laplace transforms, we have:

$$sI - A = \begin{bmatrix} \left(s + \frac{4}{5}\right) & -\frac{3}{5} \\ \frac{2}{5} & \left(s + \frac{11}{5}\right) \end{bmatrix}$$

so

$$\Phi(s) = (sI - A)^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} \left(s + \frac{11}{5}\right) & \frac{3}{5} \\ -\frac{2}{5} & \left(s + \frac{4}{5}\right) \end{bmatrix}$$

After partial fraction expansion, this becomes:

$$\Phi(s) = \begin{bmatrix} \left(\frac{\frac{6}{5}}{s+1} - \frac{\frac{1}{5}}{s+2}\right) & \left(\frac{\frac{3}{5}}{s+1} - \frac{\frac{3}{5}}{s+2}\right) \\ \left(\frac{-\frac{2}{5}}{s+1} + \frac{\frac{2}{5}}{s+2}\right) & \left(\frac{-\frac{1}{5}}{s+1} + \frac{\frac{6}{5}}{s+2}\right) \end{bmatrix}$$

Therefore,

$$e^{At} = \mathcal{L}^{-1}(\Phi(s)) = \begin{bmatrix} \left(\frac{6}{5}e^{-t} - \frac{1}{5}e^{-2t}\right) & \left(\frac{3}{5}e^{-t} - \frac{3}{5}e^{-2t}\right) \\ \left(-\frac{2}{5}e^{-t} + \frac{2}{5}e^{-2t}\right) & \left(-\frac{1}{5}e^{-t} + \frac{6}{5}e^{-2t}\right) \end{bmatrix}$$

iii) To make use of the Jordan canonical form, we first need to compute the eigenvectors of A. Fortunately, this was already done in the previous handout, yielding

$$M = \begin{bmatrix} -3 & -\frac{1}{2} \\ 1 & 1 \end{bmatrix} \quad ; \quad M^{-1} = -\frac{2}{5}\begin{bmatrix} 1 & \frac{1}{2} \\ -1 & -3 \end{bmatrix}$$

$$J = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

Since $M^{-1}AM=J$ it follows that

$$M^{-1} f(A) M = f(J) \quad \Rightarrow \quad f(A) = M f(J) M^{-1}$$

Function f(J) is easy to compute, since J is diagonal:

$$f(J) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$$

Consequently,

$$f(A) \equiv e^{At} = Mf(J)M^{-1} = \begin{bmatrix} \left(\frac{6}{5}e^{-t} - \frac{1}{5}e^{-2t}\right) & \left(\frac{3}{5}e^{-t} - \frac{3}{5}e^{-2t}\right) \\ \left(-\frac{2}{5}e^{-t} + \frac{2}{5}e^{-2t}\right) & \left(-\frac{1}{5}e^{-t} + \frac{6}{5}e^{-2t}\right) \end{bmatrix}$$

2. Here we have the same matrix A as in problem 1., but a different function: $f(x)=x^k$. Also, we don't have to do the problem in all three ways, but only using the Cayley-Hamilton theorem, and the Jordan form.

i) Following the first approach, we have:

$$f(\lambda_1) \equiv (-1)^k = \alpha_o + \alpha_1 \lambda_1$$

$$f(\lambda_2) \equiv (-2)^k = \alpha_o + \alpha_1 \lambda_2$$

resulting in system

$$\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \alpha_o \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} (-1)^k \\ (-2)^k \end{bmatrix}$$

Solving for α_o and α_1, we obtain:

$$\alpha_o = 2(-1)^k - (-2)^k \quad , \quad \alpha_1 = (-1)^k - (-2)^k$$

We now have:

$$A^k \equiv f(A) = \alpha_o I + \alpha_1 A = \begin{bmatrix} \left(\frac{6}{5}(-1)^k - \frac{1}{5}(-2)^k\right) & \left(\frac{3}{5}(-1)^k - \frac{3}{5}(-2)^k\right) \\ \left(-\frac{2}{5}(-1)^k + \frac{2}{5}(-2)^k\right) & \left(-\frac{1}{5}(-1)^k + \frac{6}{5}(-2)^k\right) \end{bmatrix}$$

ii) Matrices M, M^{-1} and J are the same as in problem 1, and

$$f(A) = Mf(J)M^{-1}$$

Again, since J is diagonal f(J) is easily computed as:

$$f(J) = \begin{bmatrix} (-1)^k & 0 \\ 0 & (-2)^k \end{bmatrix}$$

Therefore,

$$f(A) = Mf(J)M^{-1} = \begin{bmatrix} \left(\frac{6}{5}(-1)^k - \frac{1}{5}(-2)^k\right) & \left(\frac{3}{5}(-1)^k - \frac{3}{5}(-2)^k\right) \\ \left(-\frac{2}{5}(-1)^k + \frac{2}{5}(-2)^k\right) & \left(-\frac{1}{5}(-1)^k + \frac{6}{5}(-2)^k\right) \end{bmatrix}$$

3. This problem is a little different than the previous two. It is intended to illustrate how we can handle situations when the eigenvalues are complex. In particular, matrix

$$A = \begin{bmatrix} -1 & -\frac{5}{2} \\ 1 & 0 \end{bmatrix}$$

has characteristic polynomial

$$|A - \lambda I| = \lambda^2 + \lambda + \frac{5}{2}$$

with roots

$$\lambda_{1,2} = -\frac{1}{2} \pm j\frac{3}{2}$$

Although the eigenvalues are complex, we can still proceed in the same way as before:

$$f(\lambda_1) = e^{\lambda_1 t} = \alpha_o + \alpha_1 \lambda_1$$

$$f(\lambda_2) = e^{\lambda_2 t} = \alpha_o + \alpha_1 \lambda_2$$

which results in system:

$$\begin{bmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{bmatrix} \begin{bmatrix} \alpha_o \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \end{bmatrix}$$

Multiplying row 1 by -1 and adding it to row 2 , this becomes

$$\begin{bmatrix} 1 & \lambda_1 \\ 0 & (\lambda_2 - \lambda_1) \end{bmatrix} \begin{bmatrix} \alpha_o \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} - e^{\lambda_1 t} \end{bmatrix}$$

Observing that $\lambda_2-\lambda_1=-j3$, it follows that

$$\alpha_1=-\frac{1}{j3}e^{-\frac{t}{2}}\left(e^{-j\frac{3}{2}t}-e^{j\frac{3}{2}t}\right)=$$

$$=-\frac{1}{j3}e^{-\frac{t}{2}}\left(-2j\sin\frac{3}{2}t\right)=\frac{2}{3}e^{-\frac{t}{2}}\sin\frac{3}{2}t$$

Going back to the first equation, we have

$$\alpha_o=-\lambda_1\alpha_1+e^{\lambda_1 t}=-\left(-\frac{1}{2}+j\frac{3}{2}\right)\left(\frac{2}{3}e^{-\frac{t}{2}}\sin\frac{3}{2}t\right)+e^{-\frac{t}{2}}e^{j\frac{3}{2}t}$$

Multiplying out the first term and expanding the second one by Euler's formula, we obtain:

$$\alpha_o=\frac{1}{3}e^{-\frac{t}{2}}\sin\frac{3}{2}t-je^{-\frac{t}{2}}\sin\frac{3}{2}t+e^{-\frac{t}{2}}\left(\cos\frac{3}{2}t+j\sin\frac{3}{2}t\right)=$$

$$=\left(\frac{1}{3}e^{-\frac{t}{2}}\sin\frac{3}{2}+e^{-\frac{t}{2}}\cos\frac{3}{2}t\right)$$

As you can see, α_o and α_1 are *real* numbers, and we can compute

$$e^{At}=\alpha_o I+\alpha_1 A=e^{-\frac{t}{2}}\begin{bmatrix}\left(\cos\frac{3t}{2}-\frac{1}{3}\sin\frac{3t}{2}\right) & \left(-\frac{5}{3}\sin\frac{3t}{2}\right)\\ \left(\frac{2}{3}\sin\frac{3t}{2}\right) & \left(\cos\frac{3t}{2}+\frac{1}{3}\sin\frac{3t}{2}\right)\end{bmatrix}$$