

## HANDOUT #3

1. Given the simulation diagram in Fig.1, construct the corresponding state model.

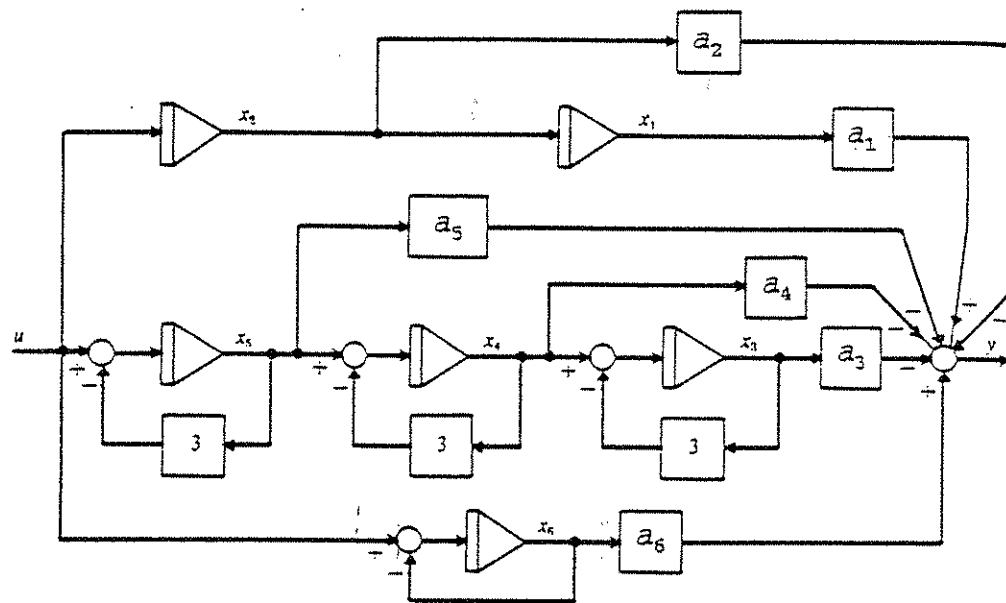


Fig.1.

2. For input-output relationship

$$y''' + 5y'' + 7y' + 3y = x' + 2x$$

form the observable, controllable and parallel state models and draw all three simulation diagrams.

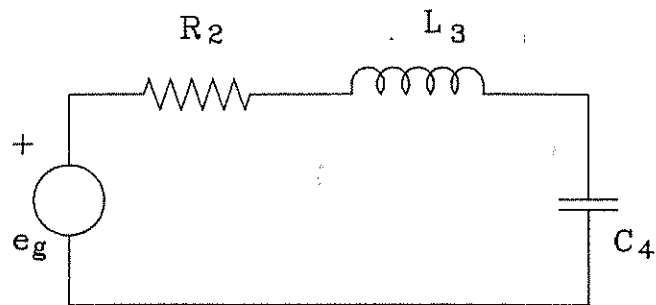
3. Given the multi-input multi-output system

$$\ddot{y}_1 + 3\dot{y}_1 + 2y_2 = u_1 + 2u_2$$

$$\ddot{y}_2 + 4\dot{y}_1 + 3y_2 = 3u_2 + u_1$$

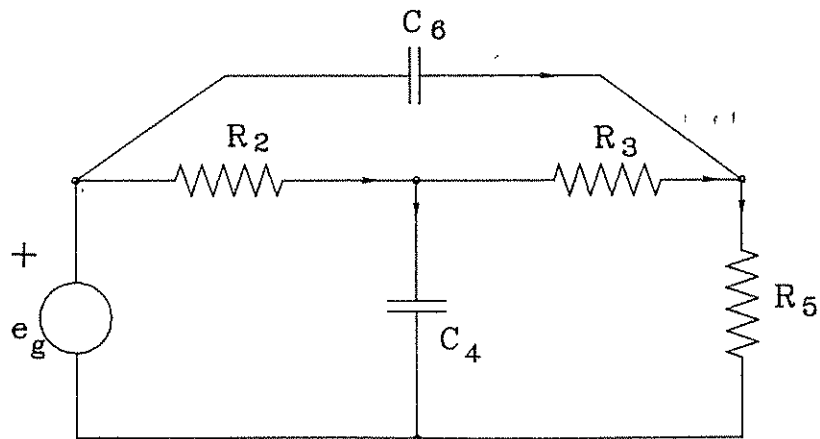
find a state model that corresponds to it.

4. Write state equations for the circuit in Fig.2.



**Fig.2.**

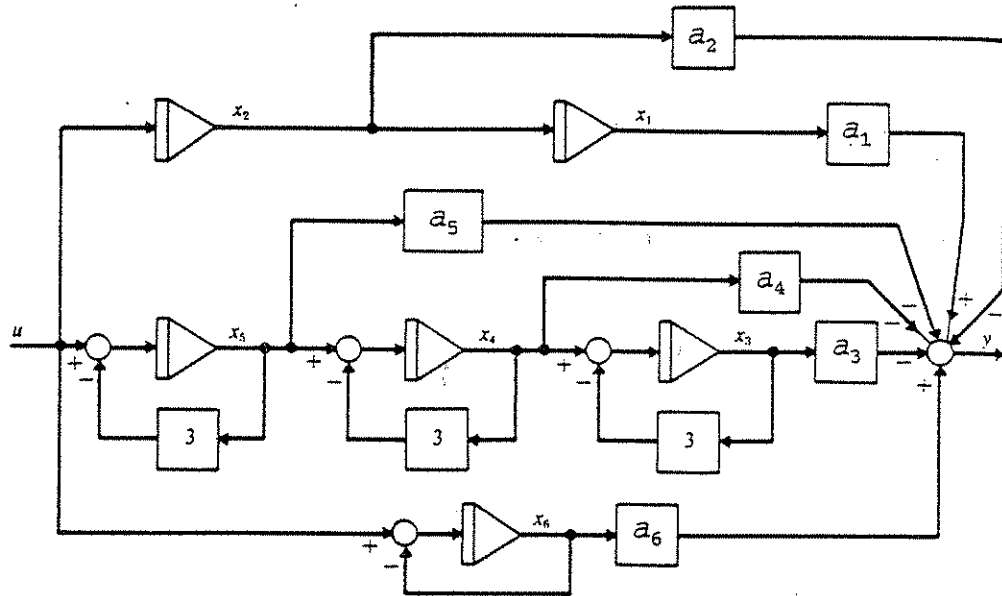
5. Write state equations for the circuit in Fig.3.



**Fig.3.**

# SOLUTIONS

1.



The states are already indicated in the block diagram, so we can write out the appropriate relationships directly. We have:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = u$$

$$\dot{x}_3 = -3x_3 + x_4$$

$$\dot{x}_4 = -3x_4 + x_5$$

$$\dot{x}_5 = -3x_5 + u$$

$$\dot{x}_6 = -x_6 + u$$

$$y = a_1x_1 - a_2x_2 - a_3x_3 - a_4x_4 - a_5x_5 + a_6x_6$$

In matrix form, the state model can be written as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = [a_1 \ -a_2 \ -a_3 \ -a_4 \ -a_5 \ a_6] \underline{x}$$

2. We are given I/O relationship:

$$y''' + 5y'' + 7y' + 3y = x' + 2x$$

By the notation we used before, we have

$$a_0 = 3 \quad ; \quad a_1 = 7 \quad ; \quad a_2 = 5$$

and

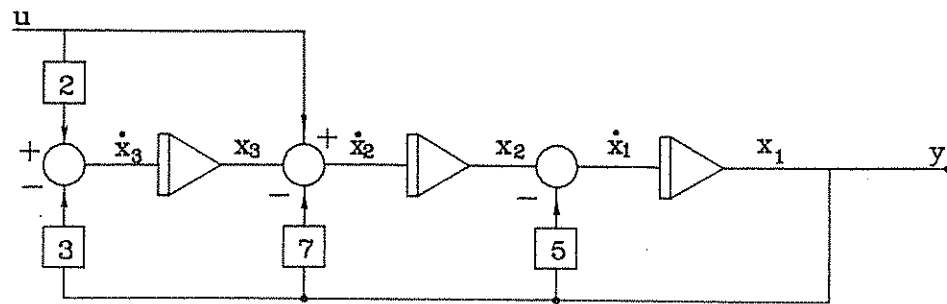
$$\beta_0 = 2 \quad ; \quad \beta_1 = 1 \quad ; \quad \beta_2 = 0$$

a) OBSERVABLE FORM

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -5 & 1 & 0 \\ -7 & 0 & 1 \\ -3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} u$$

$$y = [1 \ 0 \ 0] \underline{x}$$

We write the above representation directly, simply substituting values for  $a_i$  and  $\beta_i$  ( $i=0,1,2$ ). The corresponding simulation diagram is:

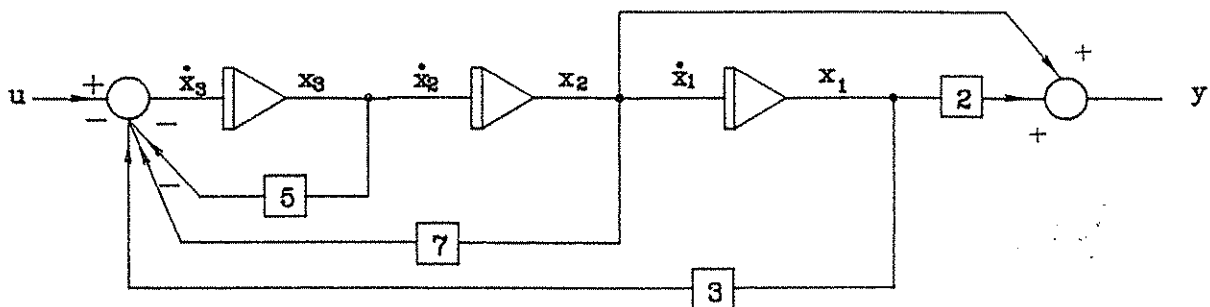


b) CONTROLLABLE FORM

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -7 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix} \mathbf{x}$$

The simulation diagram will be:



c) PARALLEL FORM

For this form, we first need the transfer function.  
In this case, taking the Laplace transform, we obtain

$$(s^3 + 5s^2 + 7s + 3)Y(s) = (s + 2)X(s)$$

Therefore,

$$G(s) = \frac{Y(s)}{X(s)} = \frac{s+2}{s^3 + 3s^2 + 7s + 3}$$

After factoring the denominator, we have:

$$G(s) = \frac{s+2}{(s+1)^2(s+3)}$$

Applying partial fraction expansion, this becomes:

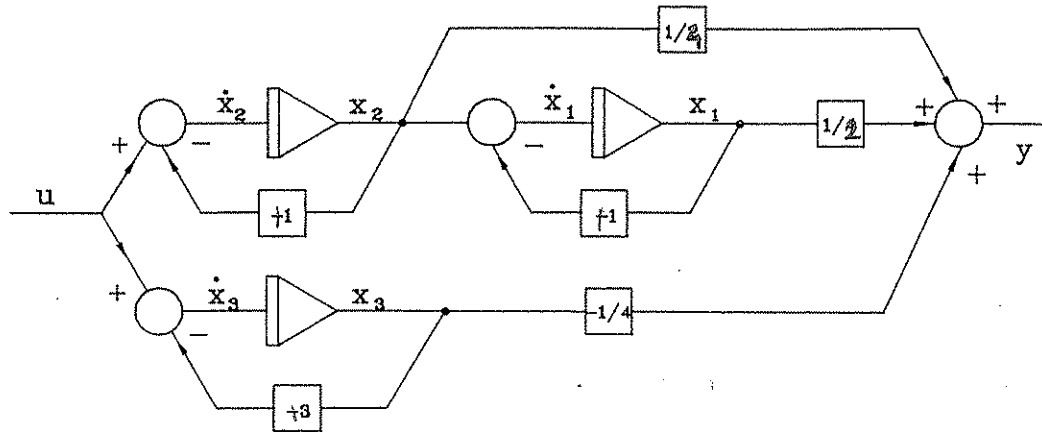
$$G(s) = \frac{\frac{1}{4}}{s+1} + \frac{\frac{1}{2}}{(s+1)^2} - \frac{\frac{1}{4}}{s+3}$$

We can now write the state model directly as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{bmatrix} \mathbf{x}$$

The corresponding simulation diagram is:



3. We are given

$$\ddot{y}_1 + 3\dot{y}_1 + 2y_2 = u_1 + 2u_2$$

$$\ddot{y}_2 + 4\dot{y}_1 + 3y_2 = 3u_2 + u_1$$

which is a multi-input, multi-output system. As we normally do in such cases, with each output we will associate state variables as:

$$\begin{aligned} x_1 &= y_1 & x_3 &= y_2 \\ x_2 &= \dot{y}_1 & x_4 &= \dot{y}_2 \end{aligned}$$

The state variables are now related as follows:

$$\dot{x}_1 = \dot{y}_1 = x_2$$

$$\dot{x}_2 = \ddot{y}_1 = -3\dot{y}_1 - 2y_2 + u_1 + 2u_2 = -3x_2 - 2x_3 + u_1 + 2u_2$$

$$\dot{x}_3 = \dot{y}_2 = x_4$$

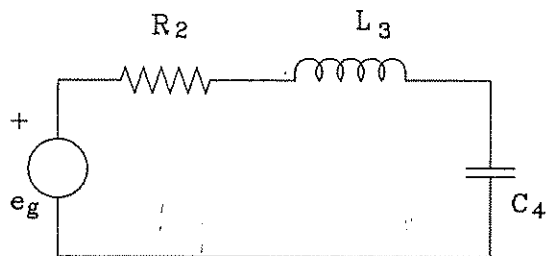
$$\dot{x}_4 = \ddot{y}_2 = -4\dot{y}_1 - 3y_2 + 3u_2 + u_1 = -4x_2 - 3x_3 + 3u_2 + u_1$$

Consequently, the state model is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -3 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -4 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

4.



In order to write the state equations for the above circuit, we will first recall that in any circuit the state variables are always the inductor currents and capacitor voltages. Since these quantities are related by Kirchoff's laws, we need to write the appropriate current and voltage law equations. We have:

$$\text{KCL:} \quad i_{C_4} = i_{L_3}$$

$$\text{KVL:} \quad V_{L_3} = e_g - V_{R_2} - V_{C_4} = e_g - R_2 i_{L_3} - V_{C_4}$$



Recalling that

$$i_{C_4} = C_4 \frac{dV_{C_4}}{dt} \quad \text{and} \quad V_{L_3} = L_3 \frac{di_{L_3}}{dt}$$

we have:

$$C_4 \frac{dV_{C_4}}{dt} = i_{L_3}$$

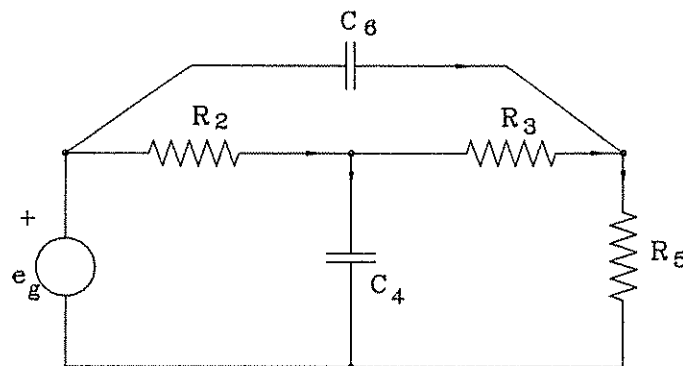
$$L_3 \frac{di_{L_3}}{dt} = e_g - R_2 i_{L_3} - V_{C_4}$$

The state model is now:

$$\begin{bmatrix} \frac{dV_{C_4}}{dt} \\ \frac{di_{L_3}}{dt} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C_4} \\ -\frac{1}{L_3} & -\frac{R_2}{L_3} \end{bmatrix} \begin{bmatrix} V_{C_4} \\ i_{L_3} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L_3} \end{bmatrix} e_g$$

(The output equation is not specified.)

5.



Now we are getting a bit more fancy. Following what we said earlier, the state variables here are the two capacitor voltages. The appropriate current and voltage law equations are given below. To simplify the notation, the currents and voltages are numbered consistently with the element they represent (eg.  $V_6 = V_{C6}$ ). Note also that here  $V_1 \equiv e_g$ . We have

$$\text{KCL: } 1) \quad i_4 + i_3 - i_2 = 0$$

$$2) \quad i_6 + i_3 - i_5 = 0$$

$$\text{KVL: } 3) \quad V_2 + V_4 - V_1 = 0$$

$$4) \quad V_5 - V_1 + V_6 = 0$$

$$5) \quad V_3 - V_6 + V_1 - V_4 = 0$$

From equation 5)  $\Rightarrow$

$$V_3 = V_6 + V_4 - e_g$$

From equation 4)  $\Rightarrow$

$$V_5 = e_g - V_6$$

From equation 3)  $\Rightarrow$

$$V_2 = e_g - V_4$$

Consequently,  $V_3$ ,  $V_5$  and  $V_2$  are now all expressed in terms of state variables and the input  $e_g$ . Observing that

$$i_3 = \frac{V_3}{R_3} \quad ; \quad i_2 = \frac{V_2}{R_2} \quad ; \quad i_5 = \frac{V_5}{R_5}$$

and substituting this into equations 1) and 2) we obtain:

$$i_4 = C_4 \frac{dV_4}{dt} = \frac{V_2}{R_2} - \frac{V_3}{R_3} = \frac{e_g - V_4}{R_2} - \frac{V_6 + V_4 - e_g}{R_3} \Rightarrow$$

$$\Rightarrow \quad \frac{dV_4}{dt} = -\left(\frac{1}{C_4 R_2} + \frac{1}{C_4 R_3}\right)V_4 - \frac{1}{C_4 R_3}V_6 + \left(\frac{1}{C_4 R_2} + \frac{1}{C_4 R_3}\right)e_g$$

and

$$i_6 = C_6 \frac{dV_6}{dt} = \frac{V_5}{R_5} - \frac{V_3}{R_3} = \frac{e_g - V_6}{R_5} - \frac{V_6 + V_4 - e_g}{R_3} \Rightarrow$$

$$\Rightarrow \frac{dV_6}{dt} = -\frac{1}{C_6 R_3} V_4 - \left( \frac{1}{C_6 R_5} + \frac{1}{C_6 R_3} \right) V_6 + \left( \frac{1}{C_6 R_5} + \frac{1}{C_6 R_3} \right) e_g$$

In matrix form, the state model can be written as:

$$\begin{bmatrix} \dot{V}_4 \\ \dot{V}_6 \end{bmatrix} = \begin{bmatrix} -\left( \frac{1}{C_4 R_2} + \frac{1}{C_4 R_3} \right) & -\frac{1}{C_4 R_3} \\ -\frac{1}{C_6 R_3} & -\left( \frac{1}{C_6 R_5} + \frac{1}{C_6 R_3} \right) \end{bmatrix} \begin{bmatrix} V_4 \\ V_6 \end{bmatrix} + \begin{bmatrix} \left( \frac{1}{C_4 R_2} + \frac{1}{C_4 R_3} \right) \\ \left( \frac{1}{C_6 R_5} + \frac{1}{C_6 R_3} \right) \end{bmatrix} e_g$$

As in the previous problem, the output was not specified.