ELEN 236

HANDOUT #1

1) Find the Laplace transform of the following functions:

a) $$e^{3t}\cos 2t$$

b) $$t\sin 2t$$

c) $$\cos(3t+45°)$$

d) $$\cos^2 t$$

2) Find the inverse Laplace transform of

$$G(s)=\frac{s+2}{s^2+2s+2}$$

3) Given a system of first order differential equations

$$\dot{y}_1=-3y_1+y_2+3x(t)$$

$$\dot{y}_2=-2y_1+2x(t)$$

where $x(t)\equiv u(t)$ and $y_1(0)=y_2(0)=1$ compute $y_1(t)$ using Laplace transforms. (Note: You don't need to compute $y_2(t)$.)

4) Given a system of first order differential equations

$$\dot{y}_1 = -\frac{1}{2} y_1 + y_2$$

$$\dot{y}_2 = -y_2 - y_1 + x(t)$$

where $x(t) \equiv t$ and $y_1(0) = y_2(0) = 0$, compute $y_2(t)$ using Laplace transforms.

SOLUTIONS

1.a) We have

$$e^{3t}\cos 2t$$

By the shift theorem,

$$F(s-3) \leftrightarrow e^{3t}f(t)$$

Since

$$f(t) = \cos 2t \leftrightarrow \frac{s}{s^2+4} \Rightarrow F(s-3) = \frac{s-3}{(s-3)^2+4}$$

Therefore,

$$\mathcal{L}(e^{3t}\cos 2t) = \frac{s-3}{(s-3)^2+4}$$

b) We have

$$t\sin 2t$$

We know that

$$(-t)^n f(t) \leftrightarrow \frac{d^n}{ds^n}F(s)$$

Setting

$$f(t)=\sin 2t \implies tf(t) \leftrightarrow -\frac{d}{ds}F(s)$$

Since

$$F(s)=\frac{2}{s^2+4} \implies -\frac{d}{ds}F(s)=\frac{4s}{(s^2+4)^2}$$

Therefore,

$$\mathcal{L}(t\sin 2t)=\frac{4s}{(s^2+4)^2}$$

c) We have

$$\cos(3t+45^\circ)$$

Therefore

$$\cos(3t+45^\circ)=\cos 3t\cos 45^\circ-\sin 3t\sin 45^\circ=\frac{1}{\sqrt{2}}(\cos 3t-\sin 3t)$$

Since

$$\mathcal{L}(\cos 3t)=\frac{s}{s^2+9} \qquad \text{and} \qquad \mathcal{L}(\sin 3t)=\frac{3}{s^2+9}$$

we obtain

$$\mathcal{L}(\cos(3t+45^\circ))=\frac{1}{\sqrt{2}}\left(\frac{s-3}{s^2+9}\right)$$

d) We have

$$\cos^2 t$$

Since

$$\cos^2 t = \frac{1}{2}(1+\cos 2t) \qquad \text{and} \qquad \mathcal{L}(\cos 2t) = \frac{s}{s^2+4}$$

it follows that

$$\mathcal{L}(\cos^2 t) = \frac{1}{2s} + \frac{1}{2}\frac{s}{(s^2+4)}$$

2. This one is easy, intended basically for refreshing your memory. We have:

$$\frac{s+2}{s^2+2s+2} = \frac{s+2}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}$$

Since

$$F(s+\alpha) \quad \leftrightarrow \quad e^{-\alpha t}f(t)$$

we directly obtain:

$$\mathcal{L}^{-1}\frac{s+2}{s^2+2s+2} = e^{-t}\cos t + e^{-t}\sin t$$

3. We have

$$\dot{y}_1 = -3y_1 + y_2 + 3x(t) \qquad y_1(0) = 1$$

$$\dot{y}_2 = -2y_1 + 2x(t) \qquad y_2(0) = 1$$

Taking the Laplace transform:

$$sy_1(s) - y_1(0) = -3y_1(s) + y_2(s) + 3x(s) \tag{1}$$

$$sy_2(s) - y_2(0) = -2y_1(s) + 2x(s) \tag{2}$$

From equation (2)

$$y_2(s)=\frac{1}{s}[-2y_1(s)+2x(s)+y_2(0)] \qquad (3)$$

Substituting into equation (1) we obtain

$$sy_1(s)-y_1(0)=-3y_1(s)+\frac{1}{s}[-2y_1(s)+2x(s)+y_2(0)]+3x(s)$$

Multiplying both sides by s, this becomes

$$(s^2+3s+2)y_1(s)=(3s+2)x(s)+sy_1(0)+y_2(0)$$

Therefore, since

$$x(s)=\frac{1}{s} \quad ; \qquad y_1(0)=y_2(0)=1$$

we have

$$y_1(s)=\frac{3s+2}{s(s+1)(s+2)}+\frac{s+1}{(s+1)(s+2)}$$

By partial fraction expansion:

$$\frac{3s+2}{s(s+1)(s+2)}=\frac{1}{s}+\frac{1}{s+1}-\frac{2}{s+2}$$

We now have;

$$y_1(s)=\left(\frac{1}{s}+\frac{1}{s+1}-\frac{2}{s+2}\right)+\frac{1}{s+2}=\frac{1}{s}+\frac{1}{s+1}-\frac{1}{s+2}$$

and consequently

$$y_1(t)=u(t)+e^{-t}-e^{-2t}$$

To compute $y_2(t)$, we could go back to equation (3) and solve $\mathcal{L}^{-1}(y_2(s))$. However, this gets a bit messy, so we might as well skip it!

4. Here we have:

$$\dot{y}_1 = -\frac{1}{2}y_1 + y_2 \qquad y_1(0) = y_2(0) = 0$$

$$\dot{y}_2 = -y_2 - y_1 + x(t)$$

Taking the Laplace transform, (and recalling $y_1(0) = y_2(0) = 0$) we have

$$sy_1(s) = -\frac{1}{2}y_1(s) + y_2(s) \qquad (1)$$

$$sy_2(s) = -y_1(s) - y_2(s) + x(s) \qquad (2)$$

From equation (2)

$$y_1(s) = -(s+1)\,y_2(s) + x(s)$$

Since by equation (1)

$$y_2(s) = \left(s + \frac{1}{2}\right)y_1(s)$$

we get

$$y_2(s)\left[\left(s+\frac{1}{2}\right)(s+1) + 1\right] = \left(s+\frac{1}{2}\right)x(s)$$

and therefore

$$y_2(s) = \frac{s+\frac{1}{2}}{s^2 + \frac{3}{2}s + \frac{3}{2}}\,x(s)$$

Since

$$x(t) = t \qquad \Rightarrow \qquad x(s) = \frac{1}{s^2}$$

We now have:

$$y_2(s) = \frac{s+\frac{1}{2}}{s^2\left(s^2+\frac{3}{2}s+\frac{3}{2}\right)}$$

Observing that

$$s^2+\frac{3}{2}s+\frac{3}{2}$$

has a pair of complex-conjugate roots, we will look for a partial fraction expansion in the form:

$$y_2(s) = \frac{A}{s}+\frac{B}{s^2}+\frac{Cs+D}{s^2+\frac{3}{2}s+\frac{3}{2}} =$$

$$= \frac{As(s^2+\frac{3}{2}s+\frac{3}{2})+B(s^2+\frac{3}{2}s+\frac{3}{2})+s^2(Cs+D)}{s^2(s^2+\frac{3}{2}s+\frac{3}{2})}$$

The numerator can be rearranged into

$$s^3(A+C)+s^2(\frac{3}{2}A+B+D)+s(\frac{3}{2}A+\frac{3}{2}B)+\frac{3}{2}B$$

and it must be equated to

$$s+\frac{1}{2}$$

Solving for A, B, C, and D (which is not as bad as it looks!), we get

$$A=\frac{1}{3} \quad ; \quad B=\frac{1}{3} \quad ; \quad C=-\frac{1}{3} \quad ; \quad D=-\frac{5}{6}$$

Consequently

$$y_2(s)=\frac{1}{3}\left\{\frac{1}{s}+\frac{1}{s^2}-\frac{s+\frac{5}{2}}{s^2+\frac{3}{2}s+\frac{3}{2}}\right\}$$

This can be rewritten as:

$$y_2(s)=\frac{1}{3}\left\{\frac{1}{s}+\frac{1}{s^2}-\frac{s+\frac{3}{4}}{\left(s+\frac{3}{4}\right)^2+\left(\frac{\sqrt{15}}{4}\right)^2}-\frac{7}{\sqrt{15}}\frac{\frac{\sqrt{15}}{4}}{\left(s+\frac{3}{4}\right)^2+\left(\frac{\sqrt{15}}{4}\right)^2}\right\}$$

and therefore

$$y_2(t)=\frac{1}{3}u(t)+\frac{1}{3}tu(t)-$$

$$-\frac{1}{3}e^{-\frac{3}{4}t}\cos\left(\frac{\sqrt{15}}{4}t\right)-\frac{7}{3\sqrt{15}}e^{-\frac{3}{4}t}\sin\left(\frac{\sqrt{15}}{4}t\right)$$