ELEN 236

HANDOUT #8

1. Show that system:

$$S:\ \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & -1\\ 2 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 2\end{bmatrix}u$$

$$y = [1 \quad 1]\ \underline{x}$$

is completely observable but *not* completely controllable. Use the appropriate transformation to identify the *uncontrollable part* of the system.

2. Show that system

$$S:\ \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & -1\\ 2 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$y = [-2 \quad 1]\ \underline{x}$$

is completely controllable but *not* completely observable. Use the appropriate transformation to identify the *unobservable part* of the system.

SOLUTIONS

1. We need to show that system

$$S:\ \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & -1\\ 2 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 2\end{bmatrix}u$$

$$y = [1 \quad 1]\,\underline{x}$$

is completely observable but not completely controllable.
For the observability test, we consider

$$Q_o = \begin{bmatrix}C\\ CA\end{bmatrix} = \begin{bmatrix}1 & 1\\ 1 & -5\end{bmatrix}$$

Clearly, $\det(Q_o)\neq 0$, so rank $Q_o=2$, and the system is completely observable.
For controllability, we consider matrix:

$$Q_c = [B \;\vdots\; AB] = \begin{bmatrix}1 & -3\\ 2 & -6\end{bmatrix}$$

for which $\det(Q_c)=0$. Multiplying row 1 by -2 and adding to row 2, Q_c is transformed into

$$\begin{bmatrix}1 & -3\\ 0 & 0\end{bmatrix}$$

Clearly, rank $Q_c=1<n$, so the system is *not* completely controllable.
To investigate which part of it is controllable, we need to transform the system into the appropriate form. To do this, we form matrix:

$$\left[\begin{array}{cc:cc} 1 & -3 & 1 & 0 \\ 2 & -6 & 0 & 1 \end{array}\right]$$

which after one elementary row transformation becomes:

$$\left[\begin{array}{cc:cc} 1 & -3 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]$$

Consequently,

$$T = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \quad \Rightarrow \quad T^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

Introducing transformation z=Tx the original system becomes:

$$\dot{z} = TAT^{-1}z + TBu$$

$$y = CT^{-1}z$$

Here,

$$\overline{A} = TAT^{-1} = \begin{bmatrix} -3 & -1 \\ 0 & -2 \end{bmatrix}$$

$$\overline{B} \equiv TB = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad ; \quad \overline{C} \equiv CT^{-1} = [3 \quad 1]$$

so we can rewrite the system explicitly as:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = [3 \quad 1]\, z$$

We now notice that equation

$$\dot{z}_2 = -2z_2$$

is *not* dependent on the input u, and therefore represents the UNCONTROLLABLE PART of the system. In this particular case, solving for z_2 yields

$$z_2 = e^{-2t} z_2(0)$$

so this variable will not cause any problems with stability.

2. This system has the same A matrix as the one in problem 1, but matrices B and C are different. Here we need to establish that the system is completely controllable but *not* completely observable.

For controllability, we consider

$$Q_c = [B \;\vdots\; AB] = \begin{bmatrix} 0 & -1 \\ 1 & -4 \end{bmatrix}$$

Clearly, $\det(Q_c) \neq 0 \Rightarrow$ rank $Q_c = 2$, so the system is indeed completely controllable.

For observability, we have

$$Q_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}$$

Here, $\det(Q_o) = 0$, and one elementary transformation on Q_o yields

$$\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}$$

Therefore, rank Q_o=1, and the system is *not* completely observable.

In order to identify the observable part, we first form the auxiliary matrix:

$$\left[\begin{array}{cc} -2 & 1 \\ 4 & -2 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right]$$

Multiplying column 1 by 1/2 and adding it to column 2 results in

$$\left[\begin{array}{c|c} -2 & 0 \\ 4 & 0 \\ \hline 1 & \frac{1}{2} \\ 0 & 1 \end{array}\right]$$

Consequently,

$$T^{-1} = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \quad \Rightarrow \quad T = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

We now introduce transformation z=Tx obtaining

$$\dot{z} = TAT^{-1} + TBu$$

$$y = CT^{-1}z$$

Since

$$TAT^{-1} = \begin{bmatrix} -2 & 0 \\ 2 & -3 \end{bmatrix}$$

$$TB = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$CT^{-1} = [-2 \quad 0]$$

we can rewrite the transformed system as:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} u$$

$$y = [-2 \quad 0]\, \underline{z}$$

We now notice that $y=-2z_1$, so the output contains information *only* about state z_1. Furthermore from equation 1

$$\dot{z}_1 = -2z_1 - \frac{1}{2}u$$

it follows that state z_1 is in no way influenced by z_2. This allows us to conclude that state z_2 is UNOBSERVABLE.