ELEN 162
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Homework 1

1. Find *all* the solutions of the following system of linear equations

$$\begin{aligned} x_1 + 3x_2 + x_3 - x_4 &= 1 \\ 2x_1 + 5x_2 + x_4 + x_5 &= 0 \end{aligned} \tag{1}$$

2. Consider the system of equations

$$\begin{aligned} x_1 + 2x_2 &= a \\ 2x_1 + x_2 &= b \\ -x_1 + x_2 &= c \end{aligned} \tag{2}$$

where a, b and c are unspecified real numbers.

   (a) What conditions do a, b and c need to satisfy in order for this system to have a solution?

   (b) Show what the solution looks like if these conditions are met (express the solution in terms of parameters a, b and c).

3. Determine whether vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 3 \\ 4 \\ 4 \\ 2 \end{bmatrix} \tag{3}$$

are linearly independent.

4. Find a basis for the nullspace of matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 3 & 1 \\ 0 & 2 & 1 & 0 & -1 \\ 0 & 1 & 1 & 1 & -1 \\ 2 & 3 & 4 & 7 & 0 \end{bmatrix} \tag{4}$$

What is the dimension of this nullspace?

5. Let S be the set of vectors that can be expressed as

$$x = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 \tag{5}$$

where α_1, α_2, α_3 and α_4 are real numbers, and

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} \tag{6}$$

When this is the case, we usually denote set S as $S = \text{span}\,\{v_1, v_2, v_3, v_4\}$, and it is not difficult to show that it has all the properties of a linear space.

(a) Determine the dimension of S.

(b) Use the Gram-Schmidt procedure to obtain an orthonormal basis for S. Show all your work!

6. If A is a symmetric, positive definite matrix with distinct eigenvalues, show that the quadratic form

$$Q(x) = x^T A x \tag{7}$$

can be bounded as

$$\lambda_m(A)\,\|x\|^2 \le x^T A x \le \lambda_M(A)\,\|x\|^2 \tag{8}$$

where $\lambda_m(A)$ and $\lambda_M(A)$ represent the smallest and largest eigenvalues of A, respectively, and $\|\cdot\|$ denotes the Euclidean norm.

7. Given matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 4 & 2 & 0 \\ 3 & 1 & 1 \end{bmatrix} \tag{9}$$

(a) Compute norms $\|A\|_2$ and $\|A\|_\infty$.

(b) Determine a vector

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{10}$$

with positive components ω_1, ω_2 and ω_3 such that the weighted infinity norm $\|A\|_\infty^\omega$ does not exceed 70% of $\|A\|_2$ or $\|A\|_\infty$. (**Hint:** To simplify the analysis, choose $\omega_1 = 1$ and $\omega_2 = \omega_3 = k$).

8. Consider matrix

$$A = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} \tag{11}$$

which is *not* symmetric. Will function

$$f(A) = A^{1/2} \tag{12}$$

be defined in this case? If so, compute it using the Jordan canonical form. If not, explain.

9. Suppose that A is a symmetric, positive definite matrix of dimension $n \times n$, with real, distinct eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.

(a) Show that $A^k e^{\alpha A}$ will be a symmetric, positive definite matrix for any integer k and real number α.

(b) Show that

$$A^k e^{\alpha A} = e^{\alpha A} A^k \tag{13}$$