Homework 2

1. Let $\hat{A}_1$, $\hat{A}_2$ and $\hat{A}_3$ be operators which are defined as

$$\hat{A}_1 f(t) = \frac{df(t)}{dt} + 3f(t) \tag{1}$$

$$\hat{A}_2 f(t) = \frac{d^2 f(t)}{dt^2} \tag{2}$$

$$\hat{A}_3 f(t) = \frac{d^2 f(t)}{dt^2} - 4k^2 t^2 f(t) \tag{3}$$

(a) Show that all three operators are *linear*.

(b) Determine the eigenvalues of operators $\hat{A}_1$ and $\hat{A}_2$ and the corresponding eigenfunctions. Is the set of eigenvalues discrete or continuous in these two cases? Explain. (**Hint:** The easiest way to solve the first two problems is to use Laplace transforms. It is also helpful to consider the positive and negative eigenvalues of $\hat{A}_2$ separately).

(c) Show that

$$f(t) = e^{-kt^2} \tag{4}$$

is an eigenfunction of operator $\hat{A}_3$, and determine the eigenvalue that corresponds to it (you can assume that k is given).

2. Show that functions $\{1, \cos x, \cos 2x, \ldots, \cos nx, \ldots\}$ are orthogonal with respect to scalar product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx \tag{5}$$

How should these functions be modified so that they become *orthonormal*?

3. Consider the sequence of functions $\{1, x, x^2, \ldots, x^n, \ldots\}$, which belong to a functional space where the scalar product is defined as

$$\langle f, g \rangle = \int_{0}^{1} f(x)g(x)dx \tag{6}$$

(a) Show that these functions are *not* orthogonal.

(b) Using set $\{1, x, x^2, \ldots, x^n, \ldots\}$ as the starting point, apply the Gram-Schmidt procedure to obtain a sequence of orthogonal polynomials $\{p_0(x), p_1(x), \ldots, p_n(x), \ldots\}$. For the purposes of this problem, it will suffice to explicitly construct only polynomials $p_0(x)$, $p_1(x)$ and $p_2(x)$. Show all your work!

(c) Are the polynomials obtained in this manner orthonormal, or just orthogonal? Explain.

4. Let S be a linear vector space in which functions ψ_0 and ψ_1 constitute an orthonormal basis, and suppose that operators $\hat{A}_0$ and $\hat{A}_1$ are defined as

$$\hat{A}_0 \psi = \langle \psi_0, \psi \rangle \psi_0 \tag{7}$$

and

$$\hat{A}_1 \psi = \langle \psi_1, \psi \rangle \psi_1 \tag{8}$$

(a) Show that ψ_0 and ψ_1 are eigenfunctions of both $\hat{A}_0$ and $\hat{A}_1$, and determine the corresponding eigenvalues.

(b) Find the matrix representations of operators $\hat{A}_0$ and $\hat{A}_1$ in basis $\{\psi_0, \psi_1\}$.

5. Let S be a linear vector space in which functions ψ_0 and ψ_1 are an orthonormal basis, and suppose that operator $\hat{A}$ transforms functions of the form

$$\psi = \alpha_0 \psi_0 + \alpha_1 \psi_1 \tag{9}$$

into

$$\hat{A}\psi = \frac{1}{\sqrt{2}}(\alpha_0 + \alpha_1)\psi_0 - \frac{1}{\sqrt{2}}(\alpha_0 - \alpha_1)\psi_1 \tag{10}$$

(a) Find the eigenfunctions of this operator, and the eigenvalues that correspond to them. **Note:** Make sure that your eigenfunctions are *normalized.*

(b) Show that the eigenfunctions you obtained in part (a) are orthogonal.

6. In this problem we will be interested in a pair of particles whose wave functions are

$$\psi_+ = \frac{1}{\sqrt{2}}\psi_0 + \frac{1}{\sqrt{2}}\psi_1 \tag{11}$$

and

$$\psi_- = \frac{1}{\sqrt{2}}\psi_0 - \frac{1}{\sqrt{2}}\psi_1 \tag{12}$$

respectively. When these two particles interact, we know that the resulting composite state will have the form

$$\Psi = \psi_+ \otimes \psi_- \tag{13}$$

where $\psi_+ \otimes \psi_-$ represents the tensor product of functions ψ_+ and ψ_-.

(a) Express state Ψ as a linear combination of basis functions Ψ_{00}, Ψ_{01}, Ψ_{10} and Ψ_{11}, and determine the probability that each of these states will be registered if a measurement is performed on both particles.

(b) Suppose we now define an operator

$$\hat{W} = \sqrt{2}\left(\hat{I} \otimes \hat{A}_0 - \hat{A}_1 \otimes \hat{I}\right) \tag{14}$$

where $\hat{I}$ is the identity operator, and $\hat{A}_0$ and $\hat{A}_1$ are the operators that were introduced in Problem 4. Find the function Φ that is obtained when $\hat{W}$ is applied to Ψ, and determine whether this function can be expressed as

$$\Phi = \psi_a \otimes \psi_b \tag{15}$$

where ψ_a and ψ_b have the general form

$$\psi_a = \alpha_0 \psi_0 + \alpha_1 \psi_1 \tag{16}$$

and

$$\psi_b = \beta_0 \psi_0 + \beta_1 \psi_1 \tag{17}$$

If so, indicate what functions ψ_a and ψ_b look like explicitly.

(c) Construct a quantum circuit that will produce state Φ using a combination of Hadamard and C-NOT gates.

(d) Repeat parts (b) and (c) in case operator $\hat{W}$ is defined as

$$\hat{W} = \sqrt{2}\left(\hat{A}_1 \otimes \hat{A}_0 - \hat{A}_0 \otimes \hat{A}_1\right) \tag{18}$$