ELEN 161
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Project 2: Strange Attractors, Chaos and Catastrophes

Problem 1. Plot the attractor for each of the four systems shown below, using the indicated initial conditions. In each case, you will have to write an m-file that generates N pairs of points $[x(n)\ y(n)]$, where N is a preassigned number. Since we are only interested in the attractor, you can discard the first 100 points (in order to eliminate transient effects).

System 1.

$$\begin{aligned} x(n+1) &= x^2(n) - y^2(n) + 0.9x(n) - 0.6013y(n) \\ y(n+1) &= 2x(n)y(n) + 2x(n) + 0.5y(n) \end{aligned} \tag{1}$$

Initial conditions: $x(0) = -0.72$; $y(0) = -0.64$.
Number of points: $N = 10^6$.

System 2.

$$\begin{aligned} x(n+1) &= 0.5\,[x(n) + x_R] \\ y(n+1) &= 0.5\,[y(n) + y_R] \end{aligned} \tag{2}$$

where pairs (x_R, y_R) are either $(0, 0)$, $(1, 0)$ or $(0.5, 1)$. The choice should be made randomly in each step (think how you can use Matlab's function randi(x) for this purpose).

Initial conditions: $x(0) = 0.5$; $y(0) = 0.5$.
Number of points: $N = 10^5$.

System 3.

$$\begin{aligned} x(n+1) &= -1.4x^2(n) + y(n) + 1 \\ y(n+1) &= 0.3x(n) \end{aligned} \tag{3}$$

Initial conditions: $x(0) = 0$; $y(0) = 0$.

Number of points: $N = 10^7$.

System 4.

$$\begin{aligned} x(n+1) &= 1 + 0.9\,[x(n)\cos r(n) - y(n)\sin r(n)] \\ y(n+1) &= 0.9\,[x(n)\sin r(n) + y(n)\cos r(n)] \end{aligned} \tag{4}$$

where

$$r(n) = 0.4 - \frac{6}{1 + x^2(n) + y^2(n)} \tag{5}$$

Initial conditions: $x(0) = 0$; $y(0) = 0$.
Number of points: $N = 10^6$.

Problem 2. Repeat Problem 1 for a class of systems of the form:

$$\begin{aligned} x(n+1) &= \alpha_1 + \alpha_2 x(n) + \alpha_3 x^2(n) + \alpha_4 x(n)y(n) + \alpha_5 y(n) + \alpha_6 y^2(n) \\ y(n+1) &= \beta_1 + \beta_2 x(n) + \beta_3 x^2(n) + \beta_4 x(n)y(n) + \beta_5 y(n) + \beta_6 y^2(n) \end{aligned} \tag{6}$$

using $x(0) = 0$, $y(0) = 0$ and $N = 10^5$ in all cases. You will need to do so for the following five choices of parameters α and β:

System 5

$$\begin{array}{llll} \alpha_1 = -0.6 & \alpha_2 = -0.1 & \alpha_3 = 1.1 & \alpha_4 = 0.2 \\ \alpha_5 = -0.8 & \alpha_6 = 0.6 & \beta_1 = -0.7 & \beta_2 = 0.7 \\ \beta_3 = 0.7 & \beta_4 = 0.3 & \beta_5 = 0.6 & \beta_6 = 0.9 \end{array}$$

System 6

$$\begin{array}{llll} \alpha_1 = -0.6 & \alpha_2 = -0.4 & \alpha_3 = -0.4 & \alpha_4 = -0.8 \\ \alpha_5 = 0.7 & \alpha_6 = 0.3 & \beta_1 = -0.4 & \beta_2 = 0.4 \\ \beta_3 = 0.5 & \beta_4 = 0.5 & \beta_5 = 0.8 & \beta_6 = -0.1 \end{array}$$

System 7

$$\begin{array}{llll} \alpha_1 = 0.8 & \alpha_2 = 1.0 & \alpha_3 = -1.2 & \alpha_4 = -1.0 \\ \alpha_5 = 1.1 & \alpha_6 = -0.9 & \beta_1 = 0.4 & \beta_2 = -0.4 \\ \beta_3 = -0.6 & \beta_4 = -0.2 & \beta_5 = -0.5 & \beta_6 = -0.7 \end{array}$$

System 8

$$\begin{array}{llll} \alpha_1 = 0.0 & \alpha_2 = -1.0 & \alpha_3 = 0.5 & \alpha_4 = -1.1 \\ \alpha_5 = -0.4 & \alpha_6 = 0.3 & \beta_1 = 0.2 & \beta_2 = 0.3 \\ \beta_3 = -0.5 & \beta_4 = 0.7 & \beta_5 = -1.1 & \beta_6 = 0.1 \end{array}$$

System 9

$$\begin{array}{llll} \alpha_1 = -0.7 & \alpha_2 = -0.4 & \alpha_3 = 0.5 & \alpha_4 = -1.0 \\ \alpha_5 = -0.9 & \alpha_6 = -0.8 & \beta_1 = 0.5 & \beta_2 = 0.5 \\ \beta_3 = 0.3 & \beta_4 = 0.9 & \beta_5 = -0.1 & \beta_6 = -0.9 \end{array}$$

Problem 3. In this problem you will examine the geometric properties of the attractors that you obtained for Systems 1-9.

(a) For each of the attractors that you computed in Problems 1 and 2, estimate the *box dimension* D_B using function boxdim.m. Because this function requires that you specify appropriate boundaries for the attractor (as $a \le x \le b$ and $a \le y \le b$), a set of suitable choices is provided in Table 1.

System 1:	$a = -2$	$b = 1$
System 2:	$a = 0$	$b = 1$
System 3:	$a = -1.5$	$b = 1.5$
System 4:	$a = -2.5$	$b = 2.5$
System 5:	$a = -1$	$b = 1$
System 6:	$a = -2.5$	$b = 1$
System 7:	$a = -1.5$	$b = 1.2$
System 8:	$a = -0.8$	$b = 1.2$
System 9:	$a = -1.2$	$b = 0.8$

Table 1. Suggested limits for x and y coordinates.

Since the results that you obtain will depend on the size of the boxes (which have dimension $l \times l$) and the number of iterations N, it is helpful to organize them in the manner shown in Table 2. In this table s is defined as $s = 1/l$, and each entry represents the output of function boxdim for a given choice of s and N.

	Parameter s								
	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}
$N = 10^5$	X	X	X	X	X	X	X	X	X
$N = 10^6$	X	X	X	X	X	X	X	X	X
$N = 10^7$	X	X	X	X	X	X	X	X	X

Table 2. Format for representing the results.

The average of each row provides an estimate for $D_B(N)$, and these estimates should converge when N becomes sufficiently large. For each of the nine systems, you will need to evaluate whether this limit has been reached for $N = 10^7$, and if so, what the corresponding value of D_B is. Since this will involve a judgment call on your part, be sure to explain your reasoning.

(b) Use function infodim.m to estimate the information dimension D_I for each of the nine systems. You should follow the same procedure as in part (a), with the understanding that the entries in the table now represent the output of function infodim for a given choice of s and N.

(c) Based on the results obtained in parts (a) and (b) of this problem, how does the information dimension compare to the box dimension? Is there a general pattern? Explain.

Problem 4. The method that you used to determine the box dimension in Problem 3 implicitly assumes that the estimated value of D_B converges when N is sufficiently large. However, not all of the systems that you were asked to analyze will reach this point for $N = 10^7$, so it may be necessary to look at larger values of N in some cases.

(a) Repeat Problem 3(a) for Systems 2, 4, 5 and 9, using $N = 10^8$ and $N = 10^9$. Note that this can take a long time for $N = 10^9$ and $s = 2^{12}$ (an hour or so), so plan accordingly.

(b) Use the data obtained in part (a) to refine your estimate of the box dimension for these four systems. In which cases does the extra computational effort pay off? And would it make sense to continue beyond $N = 10^9$? Explain.

Problem 5. Consider the system

$$\begin{aligned}\dot{x}_1 &= -x_2 - x_3 \\ \dot{x}_2 &= x_1 + qx_2 \\ \dot{x}_3 &= 0.1 + x_1x_3 - px_3\end{aligned} \tag{7}$$

where p and q are variable parameters. For certain values of p and q, this system exhibits chaotic behavior and has a strange attractor. In the following, you will be asked to determine what these values are, and to plot the attractor for one representative combination.

As you start working on this problem, you should keep in mind that computing precise bounds for p and q can be challenging in general (since it is not easy to determine when periodic behavior ends and chaos begins). One way to do this is to monitor how the solution is affected by small perturbations in the initial conditions, since hypersensitivity to such changes is one of the "trademarks" of chaos. In parts (a), (b) and (c) we will adopt this approach, since it is simple and reasonably accurate.

(a) Set $q = 0.1$ and vary p on the interval $5 \leq p \leq 20$, in steps of 0.05. Write an m-file that records the maximal difference between the two solutions

$$F(p) = \max_{t\in[0 \;\; 300]} |x(t) - y(t)| \tag{8}$$

for each choice of p. In your simulations, you can set

$$x_0 = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} \quad \text{and} \quad y_0 = \begin{bmatrix} 0.00001 \\ 5.00001 \\ 0.00001 \end{bmatrix} \tag{9}$$

as your initial conditions.

Note. In this problem, you will need to make sure that the two solutions are computed for *exactly* the same set of points, so that you can subtract them (Matlab's differential equation solver doesn't do that automatically). You can make this adjustment by using the Matlab command:

$$[t, x] = \text{ode45}(@(t, x)\ \text{function}(t, x, p),\ \text{tspan},\ x_0)$$

where tspan is a vector of preassigned points. I suggest you use tspan = $0 : 0.02 : 300$ in your calculations.

(b) Plot the function $F(p)$ obtained in part (a) and use this diagram to estimate which values of p correspond to chaotic behavior (think how the Matlab function find $(a < b)$ can help you do this accurately). For the sake of simplicity, you can assume that the behavior is periodic if $F(p) < 0.3$, and chaotic otherwise.

(c) Set $p = 5.7$ and vary q on the interval $-1 \le q \le 0.3$, in steps of 0.005. Repeat parts (a) and (b) of this problem to determine which values of q give rise to chaotic dynamics.

Note. In this case the chaotic regime is briefly "interrupted" on a narrow interval of the q-axis, where $F(q) < 0.3$ for an overwhelming majority of the points. To identify this region, you will need to "zoom" into the graph using a finer resolution for q.

(d) Pick a pair of values (p, q) for which the dynamics are clearly chaotic, and plot the strange attractor (for this purpose, you can use the Matlab function plot3(x,y,z)). Choose the same initial condition x_0 as in (9).

Problem 6. Catastrophes are characterized by abrupt changes in equilibria, which occur when one or more parameters are slightly perturbed. In the following, you will analyze this property for the system

$$\dot{x} = -x^3 + 3x - p \tag{10}$$

where p is allowed to vary continuously.

(a) Find the range of values for p for which the system has an unstable equilibrium (you should do this *analytically*, not by simulation). Denote the lower and upper bounds of this range by p_1 and p_2, respectively.

(b) Write an m-file that will produce the system equilibria as a function of p. Plot these equilibria for $-4 \le p \le 4$.

Note. Matlab has a "brush" icon in its figure menu, which allows you to erase unwanted lines. Think about how this can help you get the correct plot in a simple way.

(c) Solve equation (11) numerically for $p = p_2 - 0.001$ and $p = p_2 + 0.001$ on the interval $0 \le t \le 100$, using $x_0 = 3$ as your initial condition in both cases. Use these diagrams to explain what happens to the system when p passes through p_2.